

Solution

Seminar Decision Procedures – Homework 3

Discussed on Wednesday, 42nd July, 2016.

Exercise 3.1 **Fourier-Motzkin-Elimination**

Use Fourier-Motzkin-Elimination to determine the feasibility of the following inequalities.

$$\begin{aligned}x_1 - x_3 &\leq 0 \\x_2 - x_3 &\leq -1 \\-x_1 &\leq 1 \\-x_2 &\leq 1\end{aligned}$$

Solution: Solve for x_1

$$\begin{aligned}x_1 &\leq x_3 \\-1 &\leq x_1 \\x_2 - x_3 &\leq -1 \\-x_2 &\leq 1\end{aligned}$$

Eliminate x_1

$$\begin{aligned}-1 &\leq x_3 \\x_2 - x_3 &\leq -1 \\-x_2 &\leq 1\end{aligned}$$

Solve for x_3

$$\begin{aligned}-1 &\leq x_3 \\x_2 + 1 &\leq x_3 \\-x_2 &\leq 1\end{aligned}$$

Eliminate x_3

$$-x_2 \leq 1$$

This inequality is satisfiable and so are the original inequalities. The variable x_2 can be set to $x_2 = 1$. For x_3 there holds

$$\begin{aligned}-1 &\leq x_3 \\x_2 + 1 &\leq x_3\end{aligned}$$

Hence x_3 can be set to $x_3 = 2$. For x_1 there holds

$$\begin{aligned}x_1 &\leq x_3 \\-1 &\leq x_1\end{aligned}$$

Hence x_1 can be set to $x_1 = 2$.

Exercise 3.2 Omega Test, Dark Shadow

Prove for all integers β, b , where b does not divide β , and

$$\left\lfloor \frac{\beta}{b} \right\rfloor \leq \frac{\beta}{b} \leq \left\lfloor \frac{\beta}{b} \right\rfloor + 1$$

that

$$\frac{\beta}{b} - \left\lfloor \frac{\beta}{b} \right\rfloor \geq \frac{1}{b} . \quad (1)$$

For simplicity assume that β, b , are positive.

Hint: Use the fact that if for an integer i with $i > 0$ it holds that $i \geq 1$.

Solution: 1. b does not divide β , hence $\frac{\beta}{b} > \left\lfloor \frac{\beta}{b} \right\rfloor$, which is equivalent to

$$\frac{\beta}{b} - \left\lfloor \frac{\beta}{b} \right\rfloor > 0$$

. Multiplying both sides with b gives

$$\beta - b \left\lfloor \frac{\beta}{b} \right\rfloor > 0$$

Because β and $b \left\lfloor \frac{\beta}{b} \right\rfloor$ are integers

$$\beta - b \left\lfloor \frac{\beta}{b} \right\rfloor \geq 1 .$$

Dividing by b gives the inequality.

Exercise 3.3 Omega Test

Consider the following set of linear constraints.

$$\begin{aligned} 27 - 13y &\leq 11x \\ -10 + 9y &\leq 7x \\ 11x &\leq 45 - 13y \\ 7x &\leq 4 + 9y \end{aligned}$$

We choose to eliminate x . It can be assumed that the real shadow of the constraints contains an integer solution.

1. Write down the four inequalities that make up the dark shadow.

2. Does the dark shadow contain an integer solution?

3. Write down all grey shadows. You do not have to write down the constraints for every i explicitly plugged in, but you have to state in which range i is. You may also leave x uneliminated.

Solution: 1. The dark shadow is:

$$\begin{aligned} 11(27 - 13y) - 11(45 - 13y) &\geq (11 - 1)(11 - 1) = 100 \\ 7(27 - 13y) - 11(4 + 9y) &\geq (11 - 1)(7 - 1) = 60 \\ 11(-10 + 9y) - 7(45 - 13y) &\geq (11 - 1)(7 - 1) = 60 \\ 7(-10 + 9y) - 7(4 + 9y) &\geq (7 - 1)(7 - 1) = 36 \end{aligned}$$

Simplifying the first inequality gives:

$$\begin{aligned} -198 &\geq 100 \\ 7(27 - 13y) - 11(4 + 9y) &\geq (11 - 1)(7 - 1) = 60 \\ 11(-10 + 9y) - 7(45 - 13y) &\geq (11 - 1)(7 - 1) = 60 \\ 7(-10 + 9y) - 7(4 + 9y) &\geq (7 - 1)(7 - 1) = 36 \end{aligned}$$

2. The first inequality in the dark shadow is obviously unsatisfiable.

3. The grey shadows given by the inequality pair $27 - 13y \leq 11x$, $11x \leq 45 - 13y$:

To the original constraints the constraint $11x = 27 - 13y + i$ is added, for each i in $\{0, \dots, \lfloor ((121) - 11 - 11)/11 \rfloor\}$

The grey shadows given by the inequality pair $27 - 13y \leq 11x$, $7x \leq 4 + 9y$:
To the original constraints the constraints the constraint $11x = 27 - 13y + i$ is added , for each i in $\{0, \dots, \lfloor ((77) - 11 - 7)/7 \rfloor\}$.

The grey shadows given by the inequality pair $-10 + 9y \leq 7x$, $7x \leq 4 + 9y$:
To the original constraints the constraints the constraint $7x = -10 + 9y + i$ is added , for each i in $\{0, \dots, \lfloor ((49) - 7 - 7)/7 \rfloor\}$.

The grey shadows given by the inequality pair $-10 + 9y \leq 7x$, $11x \leq 45 - 13$:
To the original constraints the constraints the constraint $7x = -10 + 9y + i$ is added , for each i in $\{0, \dots, \lfloor ((77) - 7 - 11)/11 \rfloor\}$.