Solution

Algorithms for the Undecided – Homework 4

Discussed on Tuesday, 5th July, 2016.

Exercise 4.1 Story

Pizza baker Donald is in a pickle. His oven has caught fire and all the beautiful ingredients in his kitchen will be destroyed in the ensuing fire. His plan is to use the additional heat to abbreviate the cooking time. To do so, he gets 3 of his employees to join him. They got a deal, that promises each of them 10 pizzas of all the ones made.

Tomatoes	Cheese	Basil	Oil	Garlic	Oregano	Anchovies	Capers	Mushrooms	Parsley	
120	1960	144	700	16	180	144	112	88	96	
	Toppings									
	Margherita			2 Tomatoes, 60 Cheese, 8 Basil, 10 Oil						
Marinara			2 Tomatoes, 1 Garlic, 6 Oregano, 10 Oil							
	2 Tomatoes, 6 Anchovies, 6 Oregano, 8 Capers, 10 Oil									
Fungi			2 Tomatoes, 40 Cheese, 4 Mushrooms, 8 Parsley, 10 Oil							
	40 Cheese, 4 Mushrooms, 6 Anchovies, 10 Oil									

Donald just wants to risk the four lives, if he can use up all of the ingredients and he can keep more pizzas to himself, then he has to give away. Give a formula in Presburger arithmetic that encodes this problem.

Solution:

	2a + 2b + 2c + 2d = 120
\wedge	60a + 40d + 40e = 1960
\wedge	8a = 144
\wedge	10a + 10b + 10c + 10d + 10e = 700
\wedge	b = 16
\wedge	6b + 6c = 180
\wedge	6c + 6e = 144
\wedge	8c = 112
\wedge	4d + 4e = 88
\wedge	8d = 96
\wedge	a+b+c+d+e > 60

Exercise 4.2 Deconstruct

Find a Presburger formula φ that is represented by the DFA $A_{\varphi} = (Q, \Sigma, \delta, q_0, F)$ and provide a satisfying assignment for the free variables in decimal notation.



Solution: Because of the nature of the labels (just naturals, no tuples), we assume that the automaton was constructed from a formula that solely consisted of one atomic formula. We notice $q_t \notin \delta(Q, \Sigma)$, thus we can't assume an equation, instead we have to aim for an inequation.

We try to find upper and lower bounds to the coefficients by reversing the computation performed to find the transition function.

$$j = \left\lfloor \frac{1}{2}(k - a\zeta) \right\rfloor$$
$$\frac{1}{2} (k - a\zeta) \in [j, j + 1]$$
$$k - a\zeta \in [2j, 2(j + 1)[$$
$$-a\zeta \in [2j - k, 2(j + 1) - k[$$
$$a\zeta \in]k - 2(j + 1), k - 2j]$$
$$a\zeta \in [k - 2(j + 1) + 1, k - 2j]$$
$$a\zeta \in [k - 2j - 1, k - 2j]$$

Using these bounds we can now define a Presburger formula that specifies a.

$$\bigwedge_{\substack{\zeta \in \{0,1\}^n \\ q \in Q}} q - 2\delta(q,\zeta) - 1 \le a\zeta \wedge a\zeta \le q - 2\delta(q,\zeta)$$

For this example we yield:

$$\begin{array}{cccc} -2 \leq a_1 \wedge a_1 \leq -1 & & & & -1 \leq a_1 \wedge a_1 \leq 0 \\ \wedge & & 1 \leq a_2 \wedge a_2 \leq 2 & & & & & 2 \leq a_2 \wedge a_2 \leq 3 \\ \wedge & & & 0 \leq a_1 + a_2 \wedge a_1 + a_2 \leq 1 & & & & & 1 \leq a_1 + a_2 \wedge a_1 + a_2 \leq 2 \end{array}$$

Simplification gets us the following:

$$a_1 = -1 \wedge a_2 = 2$$

From q_0 we get b and can finally piece ϕ together:

$$\varphi = -1x_1 + 2x_2 \le -3$$

 $\{(x_1,3),(x_2,0)\}$ is one possible satisfying assignment.

Exercise 4.3 Construct

Construct automata for the formulas and provide a satisfying assignment for the free variables in decimal notation. The last one is a bonus task and will not be graded, there won't be a solution either.

$$\varphi_1 := x_1 - 2x_2 \le 3, \varphi_2 := -2x_1 - x_2 \le -4, \varphi := \varphi_1 \lor \varphi_2$$

Solution:

