

Solution Sheet – Combination of Theories

Exercise 1

Prove that the following combination of Linear Arithmetic over \mathbb{R} and Uninterpreted Functions is satisfiable using the Nelson–Oppen procedure:

$$(x_2 \geq 0) \wedge (f(x_1) = x_3) \wedge (f(x_1) \geq x_1) \wedge (x_1 - x_2 \geq x_3) \wedge (f(x_3) \geq 0)$$

- a) Show the formula after purification with optimization, where the same functions or values are assigned to the same auxiliary variable.

After purification:

$$(x_2 \geq 0) \wedge (f(x_1) = x_3) \wedge (f(x_1) = a_1) \wedge (a_1 \geq x_1) \wedge (x_1 - x_2 \geq x_3) \wedge (f(x_3) = a_2) \wedge (a_2 \geq 0)$$

- b) Show the equalities implied by the purified formula and follow the rest of the Nelson-Oppen procedure until no further equality can be implied.

F_1	F_2
$x_2 \geq 0$ $a_1 \geq x_1$ $x_1 - x_2 \geq x_3$ $a_2 \geq 0$	$f(x_1) = x_3$ $f(x_1) = a_1$ $f(x_3) = a_2$
$a_1 = x_3$ $x_2 = 0$ $x_1 = x_3$ $a_2 = x_3$	$a_1 = x_3$ $x_1 = x_3$ $a_2 = x_3$

→ Satisfiable (no error)!

Exercise 2

Prove that the following combination of three different theories (Linear Arithmetic over \mathbb{R} , Uninterpreted Functions starting with the letters a-m and Uninterpreted Functions starting with the letters n-z) is unsatisfiable using the Nelson-Oppen procedure. Therefore, use a table with three columns (F_1, F_2, F_3), one per theory.

$$(x_2 \geq x_1) \wedge (x_1 - x_3 \geq x_2) \wedge (z(f(x_1) - f(x_2)) \neq z(x_3)) \wedge (x_3 \geq 0)$$

After purification:

$$(x_2 \geq x_1) \wedge (x_1 - x_3 \geq x_2) \wedge (x_3 \geq 0) \wedge (a_1 = a_2 - a_3) \wedge (a_2 = f(x_1)) \wedge (a_3 = f(x_2)) \wedge (z(a_1) \neq z(x_3))$$

F_1	F_2	F_3
$x_2 \geq x_1$ $x_1 - x_3 \geq x_2$ $x_3 \geq 0$ $a_1 = a_2 - a_3$	$a_2 = f(x_1)$ $a_3 = f(x_2)$	$z(a_1) \neq z(x_3)$
$x_3 = 0$		

$a_2 = a_3$ $a_1 = 0$ $a_1 = x_3$	$a_2 = a_3$ $a_1 = x_3$	$a_2 = a_3$ $a_1 = x_3$ $\rightarrow \mathbf{false}$
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→ Unsatisfiable!

Exercise 3

Prove that the following combination of Linear Arithmetic over \mathbb{Z} and Uninterpreted Function is unsatisfiable using the Nelson-Oppen procedure for non-convex formulas.

$$1 \leq x \wedge x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$$

After purification:

$$(1 \leq x) \wedge (x \leq 2) \wedge (a_0 = 1) \wedge (a_1 = 2) \wedge (f(x) \neq f(a_0)) \wedge (f(x) \neq f(a_1))$$

F₁	F₂
$1 \leq x$ $x \leq 2$ $a_0 = 1$ $a_1 = 2$	$f(x) \neq f(a_0)$ $f(x) \neq f(a_1)$
$x = 1 \vee x = 2$	

F₁	F₂	F₁	F₂
$1 \leq x$ $x \leq 2$ $a_0 = 1$ $a_1 = 2$	$f(x) \neq f(a_0)$ $f(x) \neq f(a_1)$	$1 \leq x$ $x \leq 2$ $a_0 = 1$ $a_1 = 2$	$f(x) \neq f(a_0)$ $f(x) \neq f(a_1)$
$x = 1$ $a_0 = x$	$a_0 = x$ $\rightarrow \mathbf{false}$	$x = 2$ $a_1 = x$	$a_1 = x$ $\rightarrow \mathbf{false}$

→ Unsatisfiable!

Exercise 4

Prove that the following combination of Linear Arithmetic over \mathbb{Z} and Uninterpreted Function is unsatisfiable using the Nelson-Oppen procedure for non-convex formulas.

$$(1 \leq x_1) \wedge (x_2 \leq 3) \wedge (x_3 \leq x_2) \wedge (x_1 \leq 3 - x_3) \wedge (x_3 = 1) \wedge g(x_1) \wedge \neg g(2) \wedge f(x_2) \neq f(x_1) \wedge f(x_2) \neq f(3) \wedge \neg g(x_2 - x_1)$$

After purification:

$$(1 \leq x_1) \wedge (x_2 \leq 3) \wedge (x_3 \leq x_2) \wedge (x_1 \leq 3 - x_3) \wedge (x_3 = 1) \wedge g(x_1) \wedge (a_1 = 2) \wedge \neg g(a_1) \wedge f(x_2) \neq f(x_1) \wedge (a_2 = 3) \wedge f(x_2) \neq f(a_2) \wedge (a_3 = x_2 - x_1) \wedge \neg g(a_3)$$

F₁	F₂
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$x_1 \leq 3 - x_3$ $x_3 = 1$ $a_1 = 2$ $a_2 = 3$ $a_3 = x_2 - x_1$	$f(x_2) \neq f(a_2)$ $\neg g(a_3)$
$x_1 = 1 \vee x_1 = 2$	

$x_1 = 1$

$x_1 = 2$

F_1	F_2	F_1	F_2
$1 \leq x_1$ $x_2 \leq 3$ $x_3 \leq x_2$ $x_1 \leq 3 - x_3$ $x_3 = 1$ $a_1 = 2$ $a_2 = 3$ $a_3 = x_2 - x_1$	$g(x_1)$ $\neg g(a_1)$ $f(x_2) \neq f(x_1)$ $f(x_2) \neq f(a_2)$ $\neg g(a_3)$	$1 \leq x_1$ $x_2 \leq 3$ $x_3 \leq x_2$ $x_1 \leq 3 - x_3$ $x_3 = 1$ $a_1 = 2$ $a_2 = 3$ $a_3 = x_2 - x_1$	$g(x_1)$ $\neg g(a_1)$ $f(x_2) \neq f(x_1)$ $f(x_2) \neq f(a_2)$ $\neg g(a_3)$
$x_1 = 1$ $\mathbf{x_1 = x_3}$ $x_2 = 1 \vee x_2 = 2 \vee x_2 = 3$	$x_1 = x_3$	$x_1 = 2$ $\mathbf{x_1 = a_1}$	$x_1 = a_1$ \rightarrow false

$x_2 = 1$

$x_2 = 2$

$x_2 = 3$

F_1	F_2	F_1	F_2	F_1	F_2
$1 \leq x_1$ $x_2 \leq 3$ $x_3 \leq x_2$ $x_1 \leq 3 - x_3$ $x_3 = 1$ $a_1 = 2$ $a_2 = 3$ $a_3 = x_2 - x_1$	$g(x_1)$ $\neg g(a_1)$ $\mathbf{f(x_2) \neq f(x_1)}$ $\mathbf{f(x_2) \neq f(a_2)}$ $\neg g(a_3)$	$1 \leq x_1$ $x_2 \leq 3$ $x_3 \leq x_2$ $x_1 \leq 3 - x_3$ $x_3 = 1$ $a_1 = 2$ $a_2 = 3$ $a_3 = x_2 - x_1$	$g(x_1)$ $\neg g(a_1)$ $\mathbf{f(x_2) \neq f(x_1)}$ $\mathbf{f(x_2) \neq f(a_2)}$ $\neg g(a_3)$	$1 \leq x_1$ $x_2 \leq 3$ $x_3 \leq x_2$ $x_1 \leq 3 - x_3$ $x_3 = 1$ $a_1 = 2$ $a_2 = 3$ $a_3 = x_2 - x_1$	$g(x_1)$ $\neg g(a_1)$ $\mathbf{f(x_2) \neq f(x_1)}$ $\mathbf{f(x_2) \neq f(a_2)}$ $\neg g(a_3)$
$x_1 = 1$ $\mathbf{x_1 = x_3}$ $x_2 = 1$ $\mathbf{x_1 = x_2}$	$x_1 = x_3$ $x_1 = x_2$ \rightarrow false	$x_1 = 1$ $\mathbf{x_1 = x_3}$ $x_2 = 2$ $\mathbf{x_2 = a_1}$ $a_3 = 1$ $x_1 = a_3$	$x_1 = x_3$ $x_2 = a_1$ $x_1 = a_3$ \rightarrow false	$x_1 = 1$ $\mathbf{x_1 = x_3}$ $x_2 = 3$ $\mathbf{x_2 = a_2}$	$x_1 = x_3$ $x_2 = a_2$ \rightarrow false

\rightarrow Unsatisfiable!