Real Arithmetic

Max Haslbeck

Fakultät für Informatik
TU München

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Real Arithmetic

Is the theory $Th(\mathbb{R}, +, *, =, <)$ decidable?

Informally: Can we decide if a closed first order formula F in $\mathbb{R}$ using $+, *, =$ and $<$ is true?

$$\exists xyz \forall abc$$
$$(12x^5y^4a^4 - 322x^4 + 78x^3y^2 - 1034 = 0) \wedge$$
$$(2y^4 - 43z^2b^4 = 0) \wedge$$
$$(38z^3y^2 - 322zc^{19} > 0) \wedge$$
$$(123z^8 - 43x^3 + y^2 < 0)$$

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Let $v_P(a)$ be the number of sign changes in the sequence $P_0(a)$, $P_1(a)$,..., $P_K(a)$

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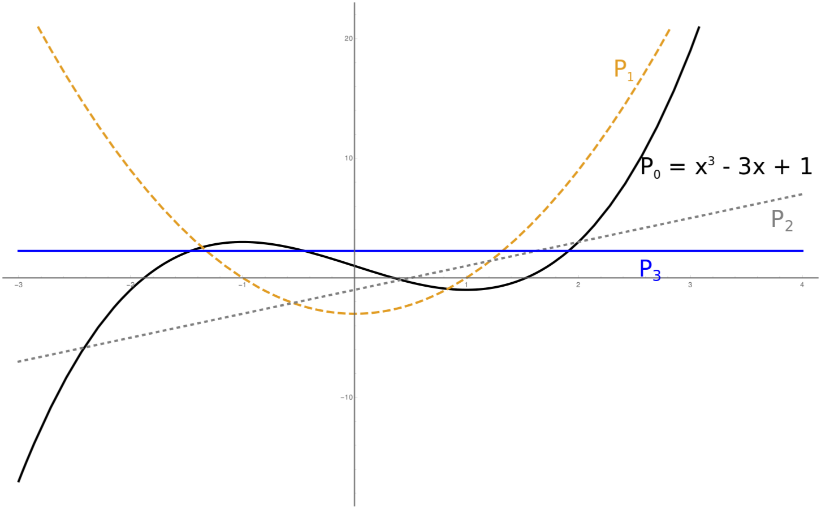
The difference $v_P(-\infty) - v_P(\infty)$ is equal to the number of *distinct* roots of P.

Example



Proof sketch

Assume P has no multiple roots:

- If c is a root of P:

x	c - ϵ	c	c + ϵ
P_0	-	0	+
P_1	+	+	+

or

x	c - ϵ	c	c + ϵ
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- For $(1 < i < K)$ if c is a root of P_i and since $P_{i-1} = P_i Q - P_{i+1}$, then $P_{i+1}(c) = -P_{i-1}(c) \neq 0$.

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Sturm sequences also work for P with multiple roots.

Multiple Equations

$$P_1 = 0, \ldots P_n = 0 \iff P_1^2 + \ldots + P_n^2 = 0$$

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Let $v(a)$ be the number of sign changes in the sequence $P_0(a)$, $P_1(a)$,..., $P_K(a)$

Lemma

Let a $<$ b be real numbers which are not roots of P.
The difference $v(a) - v(b)$ is equal to the number of distinct roots of P in the interval (a, b) such that $Q(c) > 0$ minus the number such hat $Q(c) < 0$.

One Equation + Multiple Inequalities

- $P = 0, Q_1 > 0, \ldots, Q_k > 0$
  (P is relatively prime with all Q_i)
- $\epsilon \ = (\epsilon_1, \ldots, \epsilon_k) \in \{0, 1\}^k$
- $\varphi \ = (\varphi_1, \ldots, \varphi_k) \in \{0, 1\}^k$
- $Q_\epsilon = \ Q_1^{\epsilon_1} \cdots Q_k^{\epsilon_e}$
- $s_\epsilon \ = \upsilon_{P,Q^\epsilon}(-\infty) - \upsilon_{P,Q^\epsilon}(\infty)$
- $c_\varphi = \#$ of distinct real roots c of P such that the sign of $Q_i(c)$ is $(-1)^{\varphi_i}$
- Let s (resp. c) be the vector whose coordinates are all s_ϵ (resp. c_φ)

Lemma

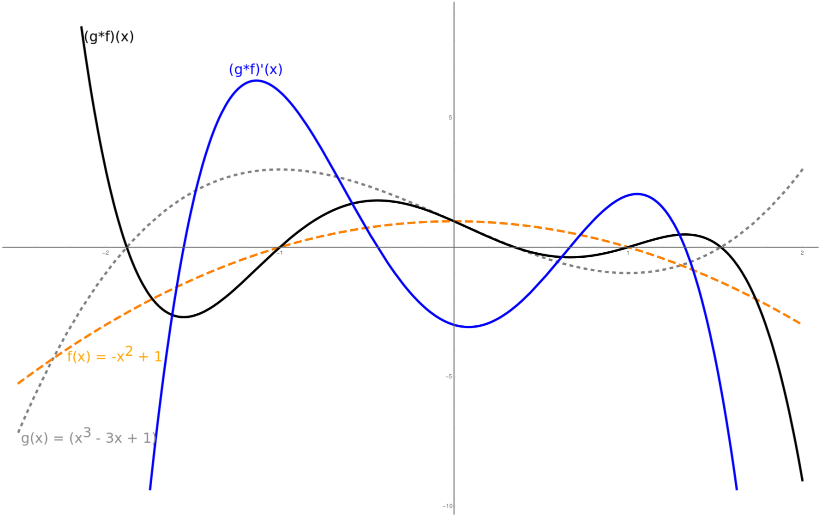
There is an invertible $2^k \times 2^k$ matrix A_k, depending only on k, such that $s = A_l \cdot c$.

Multiple inequalities

$Q_1 > 0, \ldots, Q_n > 0$

- The system is satisfied on an unbounded interval iff the leading coefficients of $Q_1, \ldots, Q_n$ or $Q_1(-X), \ldots, Q_n(-X)$ are all positive
- Let $Q = \Pi_{i=1}^{n} Q_i$. The system is satisfied iff the system $Q' = 0, Q_1 > 0, \ldots, Q_n > 0$ has a real solution.

Example



Quantifier elimination

[A. Tarski 1951, A. Seidenberg 1952]

Let $\mathcal{S}(T, X)$ be system of polynomial equations/inequalities in the variables $T = (T_1, \ldots, T_n)$ and X.

There exists a disjunction $\mathcal{C}$ of polynomial equations/equalities $\mathcal{C}_1(T) \vee \ldots \vee \mathcal{C}_n(T)$ which is equivalent to $\exists X\ \mathcal{S}(T, X)$.
$\mathcal{C}$ is computable.

Complexity and Other Methods

- Tarski's algorithm has NONELEMENTARY complexity (execution time not bound by a tower of $2^{2^{\cdot^{\cdot^{\cdot^{n}}}}}$)
- Cylindrical algebraic decomposition (George Collins, 1975)
- Worst case runtime is doubly exponential

Thank you.

Michel Coste.
An introduction to semialgebraic geometry.
RAAG network school, 145, 2002.