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Functional Data Structures

Exercise Sheet 7

Exercise 7.1 Round wrt. Binary Search Tree

The distance between two integers x and y is $|x - y|$.

1. Define a function *round* :: *int tree* ⇒ *int* ⇒ *int option*, such that *round t x* returns an element of a binary search tree t with minimum distance to x, and *None* if and only if t is empty.

   Define your function such that it does no unnecessary recursions into branches of the tree that are known to not contain a minimum distance element.
2. Specify and prove that your function is correct. Note: You are required to phrase the correctness properties yourself!

   Hint: Specify 3 properties:

   - None is returned only for the empty tree.
   - Only elements of the tree are returned.
   - The returned element has minimum distance.
3. Estimate the time of your round function to be linear in the height of the tree

fun *round* :: *"int tree ⇒ int ⇒ int option"*
fun *T_round* :: *"int tree ⇒ int ⇒ nat"*

Exercise 7.2 Interval Lists

Sets of natural numbers can be implemented as lists of intervals, where an interval is simply a pair of numbers. For example the set $\{2, 3, 5, 7, 8, 9\}$ can be represented by the list $[(2, 3), (5, 5), (7, 9)]$. A typical application is the list of free blocks of dynamically allocated memory.

We introduce the type

type_synonym *intervals* = *"(nat∗nat) list"*

Next, define an *invariant* that characterizes valid interval lists: For efficiency reasons intervals should be sorted in ascending order, the lower bound of each interval should

be less than or equal to the upper bound, and the intervals should be chosen as large as possible, i.e. no two adjacent intervals should overlap or even touch each other. It turns out to be convenient to define *inv* in terms of a more general function such that the additional argument is a lower bound for the intervals in the list:

fun *inv'* :: *"nat ⇒ intervals ⇒ bool"*
definition *inv* **where** *"inv ≡ inv' 0"*

To relate intervals back to sets define an *abstraction function*

fun *set_of* :: *"intervals ⇒ nat set"*

Define a function to add a single element to the interval list, and show its correctness

fun *add* :: *"nat ⇒ intervals ⇒ intervals"*
lemma *add_correct_1*:
"inv is ⟹ inv (add x is)"
lemma *add_correct_2*:
"inv is ⟹ set_of (add x is) = insert x (set_of is)"

Hints:

- Sketch the different cases (position of element relative to the first interval of the list) on paper first
- In one case, you will also need information about the second interval of the list. Do this case split via an auxiliary function! Otherwise, you may end up with a recursion equation of the form $f\ (x\#xs) = \ldots$ *case xs of* $x'\#xs' \Rightarrow \ldots\ f\ (x'\#xs')$ ... combined with *split*: *list.splits* this will make the simplifier loop!

Homework 7.1 Set operations on extended open intervals

Submission until Thursday, June 3, 23:59pm.

In this exercise, we are considering intervals which are open on the right side. We also allow unbounded interval, i.e. the right side can be ∞. For that, we use *enat*s, which are defined as natural numbers or ∞.

value *"[1,∞)"*

We adapt the invariant accordingly:

fun *inv'* :: *"nat ⇒ intervals ⇒ bool"* **where**
"inv' k [] = True"
| *"inv' k [[l,∞)] = (k ≤ l)"*
| *"inv' k ([l,r)#ins) = (k≤l ∧ l<r ∧ inv' (Suc r) ins)"*
| *"inv' _ _ = False"*

definition *inv* **where** *"inv = inv' 0"*

Define the set abstraction function. You can match the *enat* with two cases: ∞ and *n*::*nat*. Caution: If you only match *n* (without type), this includes ∞ – but if you use this *n* later in a function that expects a *nat* (say *Suc n*), then it will implicitly only match *nat*s.

fun *set_of* :: *"intervals ⇒ nat set"*

lemma *"set_of [[4,10), [42,∞)] = {4..9} ∪ {42..}"*
  by (*auto simp*: *numeral_eq_enat*)

We now want to build the interval list for a sorted list of *nat*s. Complete that definition:

fun *list_intvls'* :: *"nat ⇒ nat ⇒ nat list ⇒ interval list"*
definition *"list_intervals xs = (case xs of [] ⇒ [] | (x#xs) ⇒ list_intvls' x (Suc x) xs)"*

Prove correctness of *list_intervals*.

For those proofs, proceed as follows: First state the theorem in terms of *inv'*, i.e.:

lemma *list_intvls'_inv*[*simp*]: *"sorted (x#xs) ⟹ l ≤ x ⟹ inv' l (list_intvls' l (Suc x) xs)"*

Then show the theorem.

Hint: A monotonicity property on *inv'* may be useful, i.e., *inv' m ins ⟹ inv' m' ins* if $m' \leq m$

theorem *list_intervals_inv*: *"sorted (x#xs) ⟹ inv (list_intervals (x#xs))"*

theorem *list_intervals_set*: *"sorted (x#xs) ⟹ set (x#xs) = set_of (list_intervals (x#xs))"*

With these intervals, we can also define set operations that our previous definition did not allow. Define the complement for an interval list (assume that *inv* holds).

fun *compl'* :: *"nat ⇒ intervals ⇒ intervals"*
definition *compl* :: *"intervals ⇒ intervals"*

Show your complement correct:

theorem *compl_inv*[*simp*]: *"inv is ⟹ inv (compl is)"*

theorem *compl_set*: *"inv is ⟹ set_of (compl is) = −set_of is"*