Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Simon Wimmer

Lambda Calculus
Winter Term 2017/18
Solutions to Exercise Sheet 14

Exercise 1 (Combinators for Shorter Proofs)

We have introduced the combinators B and C to obtain shorter CL-translations of λ-terms. What axioms do these correspond to in the Hilbert system?

Solution

We can read these axioms from the combinators' types. For B:

$$(Q \to R) \to (P \to Q) \to P \to R$$

For C:

$$(P \to Q \to R) \to Q \to P \to R$$

Exercise 2 (Proof Translation)

Recall our proof of

$$((((P \to Q) \to P) \to P) \to Q) \to Q$$

in intuitionistic logic. Translate the proof to the Hilbert system using the axioms from the last exercise (in addition to the ones known from the lecture).

Solution

We first write down the λ-proof term:

$$\lambda x_1.\,(\lambda x_3.\,x_3\,(\lambda x_p.\,x_1\,(\lambda\_.\,x_p)))$$

One needs to remove λ-abstractions by applying λ^*, working from the inside outwards. The resulting CL-proof term is:

$$\mathsf{S}\ \mathsf{I}\ (\mathsf{B}\ (\mathsf{C}\ \mathsf{I})\ (\mathsf{C}\ \mathsf{B}\ \mathsf{K}))$$

It is important to not forget about the η-rule in the second λ^*-step, or one will get a much larger proof term.

Exercise 3 (Proof in the Hilbert System)

Prove in the Hilbert System:

$$(P \to Q) \to (R \to P) \to R \to Q$$

Solution

This is the axiom corresponding to B, so we already know a proof term via B = S (K S) K. We can turn the proof term into a Hilbert System proof by finding the right instantiations for the axioms used. This works by applying unification to each pair of theorems that are used in an application of modus ponens.

We first use the deduction rule to obtain a new theorem

$$Q_1 \to S_1 \text{ with } S_1 = (P_2 \to Q_2 \to R_2) \to (P_2 \to Q_2) \to P_2 \to R_2$$

with the proof term K S. Now we use S (K S) to deduce:

$$(Q_1 \to P_2 \to Q_2 \to R_2) \to Q_1 \to (P_2 \to Q_2) \to P_2 \to R_2$$

For the final step, we unify $Q_1 \to (P_2 \to Q_2) \to P_2 \to R_2$ with $(P \to Q) \to (R \to P) \to R \to Q$ and thus get $Q_1 = P \to Q$, $P_2 = R$, $Q_2 = P$, and $R_2 = Q$. This yields the instantiation $(P \to Q) \to R \to (P \to Q)$ for the final application of K.

Exercise 4 (Extending the System)

Suppose we want to extend the Hilbert system with $\wedge$ and $\vee$. What would the corresponding new axioms, combinators and proof translation rules look like?

Solution

The combinators would essentially still be the same as for the λ-calculus. The corresponding axioms can be deduced from their types and the translation would just recursively follow the structure of the combinators.