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Lambda Calculus
Winter Term 2021/22
Exercise Sheet 10

Exercise 1 (Intuitionistic Proofs)

Prove the following propositions in intuitionistic logic:

a) $(A \to A) \lor B$

b) $A \to (B \to A \land B)$

c) $(A \to C) \to ((B \to C) \to (A \lor B \to C))$

Exercise 2 (Intuitionistic Proof Search in Haskell)

The goal of this exercise is to implement the procedure to decide $\Gamma \vdash A$ in Haskell, i.e. the algorithm from the proof of Theorem 4.1.2.

- Have a look at the template provided on the website. It provides definitions of formulae and proof terms of intuitionistic propositional logic.
- Try to fill in the implementation of *solve*.
- Implement the three proof rules seen in the lecture: *assumption*, *intro*, and *elim*. Use the examples at the end of the template to test your implementation as you go. For *elim*, use the criterion from the proof to guess suitable instantiations.

The algorithm can be streamlined further:

a) When trying to prove $\Gamma \vdash A \to B$, it suffices to try ($\to$Intro). Explain why.

b) The attempt to prove $\Gamma \vdash A$ by assumption can be dropped if we use the following generalised $\to$Elim rule:

$$\frac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \qquad \forall i \leq n.\ \Gamma \vdash A_i}{\Gamma \vdash A \to B} \to\textsc{Elim}$$

However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.

Homework 3 (From Proof Terms to Propositions)

Consider the following proof term:

$$\lambda\ q.\ \lambda\ p.\ \ \mathsf{case}\ \pi_1\ p\ \mathsf{of}\ \mathsf{in}_1\ a \Rightarrow \mathsf{in}_1\ (\pi_1\ q,\ (a,\ \pi_2\ p))\ |\ \mathsf{in}_2\ b \Rightarrow \mathsf{in}_2\ (\pi_2\ q,\ b)$$

a) Exhibit the proposition that is proved by this term.

b) Give the corresponding proof tree.

Homework 4 (Intuitionistic Proofs)

Prove the following propositions in pure logic, without lambda-terms, and write down the λ-term corresponding to each proof:

a) $\neg(A \vee B) \to \neg A \wedge \neg B$

b) $\neg A \wedge \neg B \to \neg(A \vee B)$

Homework 5 (The Negative Fragment)

In this exercise, we consider the the fragment of intuitionistic logic where the only logical operator is $\to$. We say that a formula A is negative if atomic formulas P only occur *negated* in A, i.e. in the form $P \to \bot$ ($\neg P$ for short).

Show, by induction on A, that if A is negative, then:

$$\vdash \neg\neg A \to A$$

Hint: First show:

a) $\vdash \neg\neg\neg A \to \neg A$

b) $\vdash \neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)$

c) $\vdash (\neg\neg A \to \neg\neg B) \to (A \to \neg\neg B)$