

Exercise 1 (Confluence & Commutation)

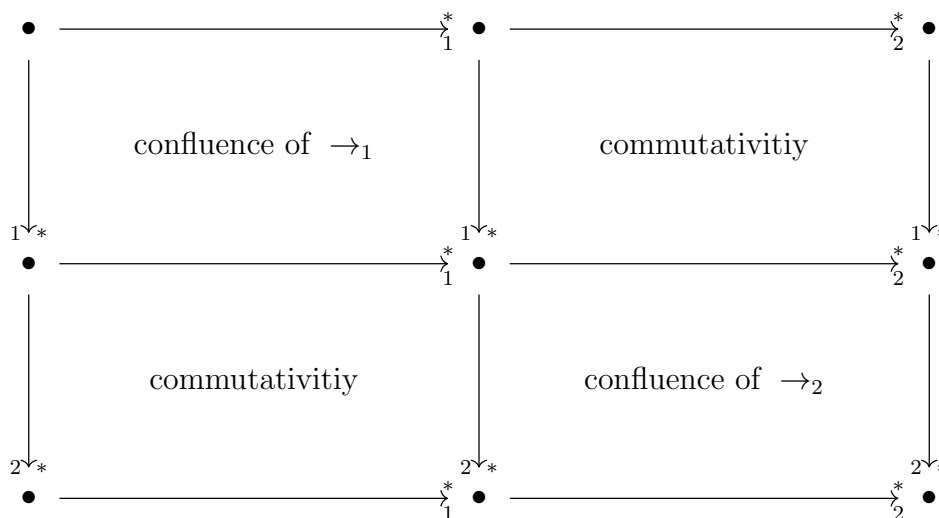
Show: If \rightarrow_1 and \rightarrow_2 are confluent, and if \rightarrow_1^* and \rightarrow_2^* commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.

Solution

Lemma A.3.2 from the lecture. The key idea is to consider $\rightarrow_1^* \circ \rightarrow_2^*$ as \rightarrow_{12}^* unfolds into iterations of this relation, i.e. $(\rightarrow_1^* \circ \rightarrow_2^*)^* = \rightarrow_{12}^*$. More precisely:

$$\rightarrow_{12} \subseteq \rightarrow_1^* \circ \rightarrow_2^* \subseteq \rightarrow_{12}^* \quad (*)$$

The relation $\rightarrow_1^* \circ \rightarrow_2^*$ has the diamond property:



With (*) and Lemma A.2.5 it immediately follows that \rightarrow_{12} is confluent.

Exercise 2 (Strong Confluence)

A relation \rightarrow is said to be *strongly confluent* iff:

$$t_2 \leftarrow s \rightarrow t_1 \implies \exists u. t_2 \rightarrow^* u \leftarrow^* t_1$$

Show that every *strongly confluent* relation is also *confluent*.

Solution

We show that every strongly confluent relation is also semi-confluent (see homework). To do so, we will show the stronger property

$$t_2 \stackrel{n}{\leftarrow} s \rightarrow t_1 \implies \exists u. t_2 \rightarrow^= u \stackrel{*}{\leftarrow} t_1$$

by induction on n . The base case for $n = 0$ is easy: $s \stackrel{0}{\leftarrow} s \rightarrow t_1 \implies s \rightarrow^= t_1 \stackrel{*}{\leftarrow} t_1$. For the induction step, we assume the statement for some u as the induction hypothesis. Furthermore, we assume $t'_2 \leftarrow t_2 \stackrel{n}{\leftarrow} s \rightarrow t_1$ for some t'_2 . We need to show that there exists a v with $t'_2 \rightarrow^= v \stackrel{*}{\leftarrow} t_1$.

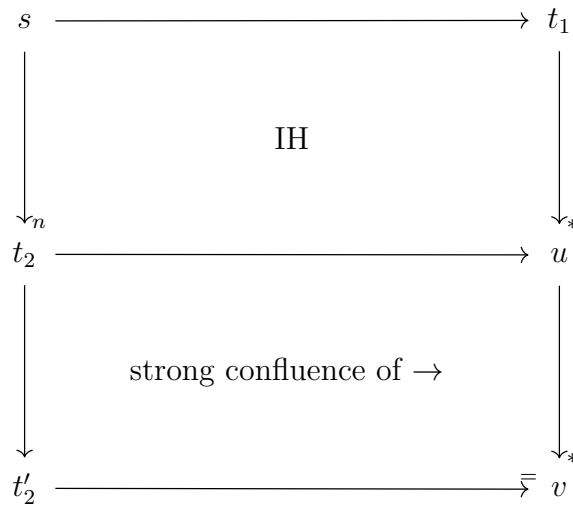
We make a case distinction on $t_2 \rightarrow^= u$.

When $t_2 = u$ we get $t_1 \rightarrow^* t_2$ with the IH and thus $t'_2 \rightarrow^= t'_2 \stackrel{*}{\leftarrow} t_1$ since $t_2 \rightarrow t'_2$.

If $t_2 \rightarrow u$, then from strong confluence of r with $t'_2 \leftarrow t_2 \rightarrow u$ we obtain a v such that

$$u \rightarrow^* v \wedge t'_2 \rightarrow^= v$$

Together with the induction hypothesis $t_1 \rightarrow^* u$, we get $t'_2 \rightarrow^= v \stackrel{*}{\leftarrow} t_1$. As a picture:



Exercise 3 (Normal Forms)

Recall the inductive definition of the set NF of *normal forms*:

$$\frac{t \in \text{NF}}{\lambda x. t \in \text{NF}}$$

$$\frac{n \geq 0 \quad t_1 \in \text{NF} \quad t_2 \in \text{NF} \quad \dots \quad t_n \in \text{NF}}{x t_1 t_2 \dots t_n \in \text{NF}}$$

Show that this set precisely captures all normal forms, i.e.:

$$t \in \text{NF} \Leftrightarrow \nexists t'. t \rightarrow_\beta t'$$

Solution

We prove the direction \implies by an induction on the derivation of $t \in \text{NF}$.

For the first case, to work towards a contradiction, we assume that $\lambda x. t \rightarrow_\beta \lambda x. t'$ and the induction hypothesis $\nexists t'. t \rightarrow_\beta t'$. By analysing the derivation of the former (\rightarrow_β), we get $t \rightarrow_\beta t'$ and thus a contradiction with the latter.

In the second case we have (IH) $\nexists t'. t_i \rightarrow_\beta t'$ for $1 \leq i \leq n$ and $n \geq 0$. We show this case by another induction on n . In the case $n = 0$ we get just x which is not a redex. Now assume $\nexists t'. x t_1 t_2 \dots t_n \rightarrow_\beta t'$ as the induction hypothesis, and $x t_1 t_2 \dots t_n t_{n+1} \rightarrow_\beta t'$ for the sake of contradiction. By analysing the derivation of the latter, we can only conclude $\exists t'. t_{n+1} \rightarrow_\beta t'$, which manifests a contradiction.

We prove the other direction indirectly by structural induction on t , i.e. we assume $t \notin \text{NF}$ and show $\exists t'. t \rightarrow_\beta t'$.

The interesting case is the application. We assume $t_1 \notin \text{NF} \implies \exists t'. t_1 \rightarrow_\beta t'$ and $t_2 \notin \text{NF} \implies \exists t'. t_2 \rightarrow_\beta t'$ as the induction hypothesis, and $t \notin \text{NF}$. The cases where $t_1 \notin \text{NF}$ or $t_2 \notin \text{NF}$ are immediate by the induction hypothesis. Consider the case $t_1, t_2 \in \text{NF}$. We analyze the derivation of $t_1 \in \text{NF}$. In the case of the rules for variables and applications, we can immediately use the rule for applications to derive the contradiction $t_1 t_2 \in \text{NF}$. Thus $t_1 = \lambda x. t'_1$ for some x, t'_1 , and we get: $t_1 t_2 \rightarrow_\beta t'_1[t_2/x]$.

Homework 4 (Semi-Confluence)

A relation \rightarrow is said to be *semi-confluent* iff:

$$t_2 \xrightarrow{*} s \rightarrow t_1 \implies \exists u. t_2 \xrightarrow{*} u \xrightarrow{*} t_1$$

Show that \rightarrow is *semi-confluent* if and only if it is *confluent*.

Homework 5 (Weak Diamond Property)

Assume that \rightarrow has the following weaker diamond property:

$$t_2 \leftarrow s \rightarrow t_1 \wedge t_1 \neq t_2 \implies \exists u. t_2 \rightarrow u \leftarrow t_1.$$

- a) Is it still the case that every element is either in normal form or has no normal form?
- b) Show that if t has a normal form, then all its reductions to its normal form have the same length.

Homework 6 (Normal Forms & Big Step)

Show:

$$t \in \text{NF} \wedge t \Rightarrow_n u \implies u = t$$