Exercise 1 (Confluence & Commutation)

Show: If $\to_1$ and $\to_2$ are confluent, and if $\to_1^*$ and $\to_2^*$ commute, then $\to_{12} := \to_1 \cup \to_2$ is also confluent.

Solution

Lemma A.3.2 from the lecture. The key idea is to consider $\to_1^* \circ \to_2^*$ as $\to_{12}^*$ unfolds into iterations of this relation, i.e. $(\to_1^* \circ \to_2^*)^* = \to_{12}^*$. More precisely:

$$\to_{12} \subseteq \to_1^* \circ \to_2^* \subseteq \to_{12}^* \qquad (*)$$

The relation $\to_1^* \circ \to_2^*$ has the diamond property:

$$\begin{tikzcd}
\bullet \arrow[r, "*"', "1"] \arrow[d, "1*"] \arrow[dr, phantom, "\text{confluence of } \to_1"] & \bullet \arrow[r, "*"', "2"] \arrow[d, "1*"] \arrow[dr, phantom, "\text{commutativitiy}"] & \bullet \arrow[d, "1*"] \\
\bullet \arrow[r, "*"', "1"] \arrow[d, "2*"] \arrow[dr, phantom, "\text{commutativitiy}"] & \bullet \arrow[r, "*"', "2"] \arrow[d, "2*"] \arrow[dr, phantom, "\text{confluence of } \to_2"] & \bullet \arrow[d, "2*"] \\
\bullet \arrow[r, "*"', "1"] & \bullet \arrow[r, "*"', "2"] & \bullet
\end{tikzcd}$$

With (*) and Lemma A.2.5 it immediately follows that $\to_{12}$ is confluent.

Exercise 2 (Strong Confluence)

A relation $\to$ is said to be *strongly confluent* iff:

$$t_2 \leftarrow s \to t_1 \Longrightarrow \exists u.\ t_2 \to^= u \ {}^*\!\!\leftarrow t_1$$

Show that every *strongly confluent* relation is also *confluent*.

Solution

We show that every strongly confluent relation is also semi-confluent (see homework). To do so, we will show the stronger property

$$t_2 \;{}^{n}{\leftarrow}\; s \to t_1 \Longrightarrow \exists u.\ t_2 \to^{=} u \;{}^{*}{\leftarrow}\; t_1$$

by induction on n. The base case for $n = 0$ is easy: $s \;{}^{0}{\leftarrow}\; s \to t_1 \Longrightarrow s \to^{=} t_1 \;{}^{*}{\leftarrow}\; t_1$. For the induction step, we assume the statement for some u as the induction hypothesis. Furthermore, we assume $t_2' \leftarrow t_2 \;{}^{n}{\leftarrow}\; s \to t_1$ for some t_2'. We need to show that there exists a v with $t_2' \to^{=} v \;{}^{*}{\leftarrow}\; t_1$.

We make a case distinction on $t_2 \to^{=} u$.

When $t_2 = u$ we get $t_1 \to^{*} t_2$ with the IH and thus $t_2' \to^{=} t_2' \;{}^{*}{\leftarrow}\; t_1$ since $t_2 \to t_2'$.

If $t_2 \to u$, then from strong confluence of r with $t_2' \leftarrow t_2 \to u$ we obtain a v such that

$$u \to^{*} v \wedge t_2' \to^{=} v$$

Together with the induction hypothesis $t_1 \to^{*} u$, we get $t_2' \to^{=} v \;{}^{*}{\leftarrow}\; t_1$. As a picture:

$$\begin{tikzcd}
s \arrow[r] \arrow[d, "n"'] \arrow[dr, phantom, "\text{IH}"] & t_1 \arrow[d, "*"] \\
t_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{strong confluence of } \to"] & u \arrow[d, "*"] \\
t_2' \arrow[r, "="] & v
\end{tikzcd}$$

Exercise 3 (Normal Forms)

Recall the inductive definition of the set NF of *normal forms*:

$$\frac{t \in \mathrm{NF}}{\lambda x.\ t \in \mathrm{NF}}$$

$$\frac{n \geq 0 \qquad t_1 \in \mathrm{NF} \qquad t_2 \in \mathrm{NF} \qquad \dots \qquad t_n \in \mathrm{NF}}{x\ t_1\ t_2\ \dots\ t_n \in \mathrm{NF}}$$

Show that this set precisely captures all normal forms, i.e.:

$$t \in \mathrm{NF} \Leftrightarrow \nexists t'.\ t \to_{\beta} t'$$

Solution

We prove the direction $\Longrightarrow$ by an induction on the derivation of $t \in \mathrm{NF}$.

For the first case, to work towards a contradiction, we assume that $\lambda x.\ t \rightarrow_\beta \lambda x.\ t'$ and the induction hypothesis $\nexists t'.\ t \rightarrow_\beta t'$. By analysing the derivation of the former ($\rightarrow_\beta$), we get $t \rightarrow_\beta t'$ and thus a contradiction with the latter.

In the second case we have (IH) $\nexists t'.\ t_i \rightarrow_\beta t'$ for $1 \leq i \leq n$ and $n \geq 0$. We show this case by another induction on n. In the case $n = 0$ we get just x which is not a redex. Now assume $\nexists t'.\ x\ t_1\ t_2\ \ldots\ t_n \rightarrow_\beta t'$ as the induction hypothesis, and $x\ t_1\ t_2\ \ldots\ t_n\ t_{n+1} \rightarrow_\beta t'$ for the sake of contradiction. By analysing the derivation of the latter, we can only conlude $\exists t'.\ t_{n+1} \rightarrow_\beta t'$, which manifests a contradiction.

We prove the other direction indirectly by structural induction on t, i.e. we assume $t \notin \mathrm{NF}$ and show $\exists t'.\ t \rightarrow_\beta t'$.

The interesting case is the application. We assume $t_1 \notin \mathrm{NF} \Longrightarrow \exists t'.\ t_1 \rightarrow_\beta t'$ and $t_2 \notin \mathrm{NF} \Longrightarrow \exists t'.\ t_2 \rightarrow_\beta t'$ as the induction hypothesis, and $t \notin \mathrm{NF}$. The cases where $t_1 \notin \mathrm{NF}$ or $t_2 \notin \mathrm{NF}$ are immediate by the induction hypothesis. Consider the case $t_1, t_2 \in \mathrm{NF}$. We analyze the derivation of $t_1 \in \mathrm{NF}$. In the case of the rules for variables and applications, we can immediately use the rule for applications to derive the contradiction $t_1\ t_2 \in \mathrm{NF}$. Thus $t_1 = \lambda x.\ t_1'$ for some x, t_1', and we get: $t_1\ t_2 \rightarrow_\beta t_1'[t_2/x]$.

Homework 4 (Semi-Confluence)

A relation $\to$ is said to be *semi-confluent* iff:

$$t_2 \ {}^*\!\!\leftarrow s \to t_1 \implies \exists u.\ t_2 \to^* u \ {}^*\!\!\leftarrow t_1$$

Show that $\to$ is *semi-confluent* if and only if it is *confluent*.

Homework 5 (Weak Diamond Property)

Assume that $\to$ has the following weaker diamond property:

$$t_2 \leftarrow s \to t_1 \wedge t_1 \neq t_2 \implies \exists u.\ t_2 \to u \leftarrow t_1.$$

a) Is it still the case that every element is either in normal form or has no normal form?

b) Show that if t has a normal form, then all its reductions to its normal form have the same length.

Homework 6 (Normal Forms & Big Step)

Show:

$$t \in \mathrm{NF} \wedge t \Rightarrow_n u \implies u = t$$