

### Exercise 1 (Confluence of $\beta$ -Reduction)

In the lecture we have shown the confluence of  $\rightarrow_\beta$  using the diamond property of parallel  $\beta$ -reduction. In this exercise, we develop an alternative proof.

We define the operation  $-^*$  on  $\lambda$ -terms inductively over the structure of terms:

$$\begin{aligned} x^* &= x \\ (\lambda x. t)^* &= \lambda x. t^* \\ (t_1 t_2)^* &= t_1^* t_2^* \quad \text{if } t_1 t_2 \text{ is not a } \beta\text{-redex.} \\ ((\lambda x. t_1) t_2)^* &= t_1^*[t_2^*/x] \end{aligned}$$

- a) Show that we have for two arbitrary  $\lambda$ -terms  $s$  and  $t$ :  $s > t \implies t > s^*$
- b) Show that  $\rightarrow_\beta$  is confluent.

### Solution

- a) The proof of

$$s > t \implies t > s^*$$

is done by  $-^*$  computation induction on  $s$  for arbitrary  $t$ . In the following, we will skip instances of  $s > s$ , as we can easily prove  $s > s^*$  by another structural induction on  $s$ .

**1st case:**  $s = x$ .

Assume  $x > t$ . Hence  $t = x$ . Thus  $t = x > x = x^* = s^*$ .

**2nd case:**  $s = \lambda x. s_1$ .

Assume  $\lambda x. s_1 > t$ . Furthermore, we get the induction hypothesis  $s_1 > t \implies t > s_1^*$  for arbitrary  $t$ . By case analysis on the derivation of  $s > t$  we obtain  $t = \lambda x. t_1$  for an appropriate  $t_1$  and we have  $s_1 > t_1$ . This implies  $t_1 > s_1^*$  considering the induction hypothesis. Thus we get  $t = \lambda x. t_1 > \lambda x. s_1^* = (\lambda x. s_1)^* = s^*$ .

**3rd case:**  $s = s_1 s_2$ , but  $s$  is not a  $\beta$ -redex.

Assume  $s_1 s_2 > t$ . As induction hypotheses we obtain  $s_i > t \implies t > s_i^*$  for arbitrary  $t$  and  $i \in \{1, 2\}$ . Then  $t = t_1 t_2$  with  $s_1 > t_1$  and  $s_2 > t_2$  (again this follows by case analysis using the inductive definition of  $>$ ). We get the induction hypotheses  $t_1 > s_1^*$  and  $t_2 > s_2^*$ . By the definition of  $>$  this implies  $t = t_1 t_2 > s_1^* s_2^* = (s_1 s_2)^* = s^*$ .

**4th case:**  $s = (\lambda x. s_1) s_2$ .

Assume  $(\lambda x. s_1) s_2 > t$ . Then there is  $t_1, t_2$  with  $s_1 > t_1$  and  $s_2 > t_2$  and we have  $t = (\lambda x. t_1) t_2$  or  $t = t_1[t_2/x]$ , depending on the rule used to derive  $s > t$ . Note that the first case actually requires a nested case distinction: first we consider the case that  $t = t'_1 t_2$  by the third rule in the definition of  $>$ . This requires that  $(\lambda x. s_1) > t'_1$ . Hence,  $t'_1 = (\lambda x. t_1)$  and  $s_1 > t_1$  follows by another case distinction on  $(\lambda x. s_1) > t'_1$ .

**Case 4.1**  $t = (\lambda x. t_1) t_2$ .

By the induction hypotheses  $t_1 > s_1^*$  and  $t_2 > s_2^*$ , we get  $t = (\lambda x. t_1) t_2 > s_1^*[s_2^*/x] = ((\lambda x. s_1) s_2)^* = s^*$ .

**Case 4.2**  $t = t_1[t_2/x]$ .

By the induction hypotheses  $t_1 > s_1^*$  and  $t_2 > s_2^*$ , and the substitution property of  $>$  (Lemma 1.2.13), we get  $t = t_1[t_2/x] > s_1^*[s_2^*/x] = ((\lambda x. s_1) s_2)^* = s^*$ .

A different proof is by structural induction on  $s$  for arbitrary  $t$ .

**Case**  $s = x$ .

Assume  $x > t$ . Hence  $t = x$  by case analysis on the  $x > t$ . Thus  $t = x > x = x^* = s^*$ .

**Case**  $s = \lambda x. s_1$ .

Assume  $\lambda x. s_1 > t$ . Furthermore, we get the induction hypothesis  $s_1 > t \implies t > s_1^*$  for arbitrary  $t$ . By case analysis on the derivation of  $s > t$  we obtain  $t = \lambda x. t_1$  for an appropriate  $t_1$  and we have  $s_1 > t_1$ . This, together with the induction hypothesis, implies  $t_1 > s_1^*$ . Thus, we get  $t = \lambda x. t_1 > \lambda x. s_1^* = (\lambda x. s_1)^* = s^*$ .

**Case**  $s = s_1 s_2$ .

Assume  $s_1 s_2 > t$ . As induction hypotheses we obtain  $s_i > t \implies t > s_i^*$  for arbitrary  $t$  and  $i \in \{1, 2\}$ . Case analysis on the derivation of  $s_1 s_2 > t$  yields two cases.

- We have  $t = t_1 t_2$  and  $s_1 > t_1$  as well as  $s_2 > t_2$ , or
- $s_1 = (\lambda x. s_3)$  and  $t = t_1[t_2/x]$  as well as  $s_3 > t_1$  and  $s_2 > t_2$ .

**Case**  $t = t_1 t_2$ .

From the induction hypotheses we have  $t_1 > s_1^*$  and  $t_2 > s_2^*$ . This implies that  $t = t_1 t_2 > s_1^* s_2^* = (s_1 s_2)^* = s^*$

**Case**  $t = t_1[t_2/x]$ .

We can only discharge the assumption of the second induction hypothesis which gives us  $t_2 > s_2^*$ . To obtain  $t_1 > s_3^*$ , we actually need an induction on the size of the term  $s$  — or a strong induction on the structure of  $s$  — such that the I.H. holds for all proper subterms of  $s$  (in particular  $s_3$ ). Together with the substitution property of  $>$  (Lemma 1.2.13), we get  $t = t_1[t_2/x] > s_3^*[s_2^*/x] = ((\lambda x. s_3) s_2)^* = s^*$ .

- b) Part a) immediately implies the diamond property of  $>$ : Let  $s > t_1$  and  $s > t_2$ . Then we have both,  $t_1 > s^*$  and  $t_2 > s^*$ . Like in the lecture we obtain the confluence of  $\rightarrow_\beta$  using the diamond property.

## Exercise 2 (Parallel Beta Reduction)

Show:

$$s > t \implies s \rightarrow_\beta^* t$$

### Solution

Proof by rule induction on  $>$ .

**Case**  $s > s$ : We have that  $s = t$  and  $s \rightarrow_\beta^* s$ .

**Case**  $\lambda x. s > \lambda x. s'$ : As an induction hypothesis we get  $s \rightarrow_\beta^* s'$ . By Lemma 1.2.3 it then holds that  $\lambda x. s \rightarrow_\beta^* \lambda x. s'$ . Consequently, we have  $\lambda x. s \rightarrow_\beta^* \lambda x. s'$ .

**Case**  $s t > s' t'$ : As induction hypotheses we get  $s \rightarrow_\beta^* s'$  and  $t \rightarrow_\beta^* t'$ . With Lemma 1.2.3 we get  $s t \rightarrow_\beta^* s' t$  and  $s' t \rightarrow_\beta^* s' t'$  and therefore  $s t \rightarrow_\beta^* s' t'$ .

**Case**  $(\lambda x. s) t > s'[t'/x]$ : As induction hypotheses we get  $s \rightarrow_\beta^* s'$  and  $t \rightarrow_\beta^* t'$ . With Lemma 1.2.3 it holds that  $(\lambda x. s) t \rightarrow_\beta^* (\lambda x. s') t \rightarrow_\beta^* (\lambda x. s') t' \rightarrow_\beta^* s'[t'/x]$  and thus  $(\lambda x. s) t \rightarrow_\beta^* (\lambda x. s') t'$ .

## Exercise 3 (Predecessor and Tail)

- Define a predecessor function `pred` on church numerals.
- Use the same idea to define `tl` on the fold encoding for lists.

### Solution

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zz = pair 0 0
ss = λp. pair (snd p) (succ (snd p))
pred = λm. fst (m ss zz)

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b)

$$\begin{aligned} \text{nxt} &= \lambda c. \lambda x p. \text{pair } (\text{snd } p) (c x (\text{snd } p)) \\ \text{tl} &= \lambda l. \lambda c n. \text{fst } (l (\text{nxt } c) (\text{pair } n n)) \end{aligned}$$

#### Homework 4 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \implies s[t/x] > s'[t'/x]$$

#### Homework 5 (Equivalence modulo $\beta$ -conversion)

Assume that we add the additional axiom

$$\lambda x y. x =_{\beta} \lambda x y. y$$

.

- a) Show that under this assumption  $t =_{\beta} t'$  for all  $t, t'$ .
- b) Repeat the same for the axiom  $\lambda x. x =_{\beta} \lambda x y. y x$ .

#### Homework 6 (Böhm's Theorem)

Böhm's Theorem states that for arbitrary closed terms  $M \neq N$  without constant atoms in  $\beta\eta$ -normal form, there exist  $n \geq 0$  and  $L_1, \dots, L_n$  such that:

$$M L_1 \dots L_n x y \rightarrow_{\beta}^* x \text{ and } N L_1 \dots L_n x y \rightarrow_{\beta}^* y.$$

That is, we can tell  $M$  and  $N$  apart. Show the following two special cases:

- a)  $M = \lambda x y z. x z (y z)$  and  $N = \lambda x y z. x (y z)$
- b)  $M = \lambda x y. x (y y)$  and  $N = \lambda x y. x (y x)$