Exercise 1 (Intuitionistic Proof Search)

The algorithm in Theorem 4.1.4 can be streamlined as follows:

a) When trying to prove $\Gamma \vdash A \to B$, it suffices to try (→Intro). Explain why.

b) The attempt to prove $\Gamma \vdash A$ by assumption can be dropped: it is subsumed by the alternative using Lemma 4.1.2. However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.

c) How would the Haskell code from the last tutorial need to be adopted to account for these improvements?

Solution

In the following we will denote the by (→Elim) the more general rule described in lemma 4.1.2.

a) Suppose we prove $\Gamma \vdash A \to B$ by an application of (→Elim). The proof will be of the following format:

$$\frac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \qquad \forall i \leq n.\ \Gamma \vdash A_i}{\Gamma \vdash A \to B}\ \to\textsc{Elim}$$

We can always provide an alternative proof that uses (→Intro) first and looks like this:

$$\frac{\dfrac{\Gamma \vdash A_1 \to \ldots \to A_n \to A \to B \qquad \forall i \leq n.\ \Gamma, A \vdash A_i \qquad \Gamma, A \vdash A}{\Gamma, A \vdash B}\ \to\textsc{Elim}}{\Gamma \vdash A \to B}\ \to\textsc{Intro}$$

The case where $\Gamma \vdash A \to B$ is proved by assumption is subsumed by the next answer.

b) Proof by assumption is just a special case of (→Elim) where $n = 0$. However, if we drop the assumption rule, proofs can now have a slightly different structure because we try (→Intro) first:

$$\frac{\dfrac{A_1 \to \ldots \to A_n \to B \in \Gamma' \qquad \forall i \leq n.\ \Gamma' \vdash A_i}{\Gamma' \vdash B}\ \to\textsc{Elim}}{\Gamma, A_1 \to \ldots \to A_n \to B \vdash A_1 \to \ldots \to A_n \to B}\ \to\textsc{Intro}\ n\ \textsc{times}$$

with

$$\Gamma' := \Gamma, A_1 \to \ldots \to A_n \to B, A_1, \ldots, A_n\,.$$

Exercise 2 (Intuitionistic Proofs)

Prove the following propositions in intuitionistic logic:

a) $(A \to A) \lor B$

b) $A \to (B \to A \land B)$

c) $(A \to C) \to ((B \to C) \to (A \lor B \to C))$

Solution

a) Term: $\mathsf{Inl}\ (\lambda x.\ x)$. Proof:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A \to A}\ {\to}\mathrm{I}}{\vdash (A \to A) \lor B}\ {\lor}\mathrm{I}_1$$

b) Term: $\lambda x\, y.\ \langle x, y\rangle$. Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash A} \qquad \dfrac{}{A, B \vdash B}}{A, B \vdash A \land B}\ {\land}\mathrm{I}}{A \vdash B \to (A \land B)}\ {\to}\mathrm{I}}{\vdash A \to B \to (A \land B)}\ {\to}\mathrm{I}$$

c) Term: $\lambda x\, y\, z.\ \mathsf{case}\, z\, \mathsf{of}\, \mathsf{Inl}\, a \Rightarrow x\, a \,|\, \mathsf{Inr}\, b \Rightarrow y\, b$. Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A \lor B} \qquad \dfrac{\dfrac{}{\Gamma, A \vdash A \to C} \qquad \dfrac{}{\Gamma, A \vdash A}}{\Gamma, A \vdash C}\ {\to}\mathrm{E} \qquad \dfrac{\dfrac{}{\Gamma, B \vdash B \to C} \qquad \dfrac{}{\Gamma, B \vdash B}}{\Gamma, B \vdash C}\ {\to}\mathrm{E}}{\Gamma := (A \to C), (B \to C), A \lor B \vdash C}\ {\lor}\mathrm{E}}{(A \to C), (B \to C) \vdash (A \lor B) \to C}\ {\to}\mathrm{I}}{(A \to C) \vdash (B \to C) \to (A \lor B) \to C}\ {\to}\mathrm{I}}{\vdash (A \to C) \to (B \to C) \to (A \lor B) \to C}\ {\to}\mathrm{I}$$

Homework 3 (Weak Normalization with Pairs)

We previously proved (sheet eight, ex. two) that every type-correct $\lambda^{\rightarrow}$-term has a β-normal form. Adapt the proof to accomodate for the extension of the simply typed lambda calculus with pairs.

Homework 4 (From Proof Terms to Propositions)

Consider the following proof term:

$$\lambda q.\ \lambda p.\ \mathsf{case}\ \pi_1\ p\ \mathsf{of}\ \mathsf{Inl}\ a \Rightarrow \mathsf{Inl}\ (\pi_1\ q,\ (a,\ \pi_2\ p))\ |\ \mathsf{Inr}\ b \Rightarrow \mathsf{Inr}\ (\pi_2\ q,\ b)$$

a) Exhibit the proposition that is proved by this term.

b) Give the corresponding proof tree.

Homework 5 (Intuitionistic Proofs)

Prove the following propositions in pure logic, without lambda-terms, and write down the λ-term corresponding to each proof:

a) $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$

b) $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$