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Lambda Calculus
Winter Term 2023/24
Exercise Sheet 2

Exercise 1 (β-reduction)

A term t is in β-normal form if there is no term t' such that $t \to_\beta t'$. List all terms t such that:

$$(\lambda x.\ (\lambda x\, y.\ x)\ z\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)\ ((\lambda x\, y.\ x)\ y)) \to_\beta^* t$$

Which are normal forms?

Exercise 2 (Lists in λ-calculus)

Specify λ-terms for nil, cons, hd, tl and null, that encode lists in the λ-calculus. Show that your terms satisfy the following conditions:

$$\begin{array}{llllll} \mathsf{null}\ \mathsf{nil} & \to_\beta^* & \mathsf{true} & \mathsf{hd}\ (\mathsf{cons}\ x\ l) & \to_\beta^* & x \\ \mathsf{null}\ (\mathsf{cons}\ x\ l) & \to_\beta^* & \mathsf{false} & \mathsf{tl}\ (\mathsf{cons}\ x\ l) & \to_\beta^* & l \end{array}$$

Hint: Use pairs.

Homework 3 (Substitution Lemma)

Show that, given $x \neq y$ and $x \notin \mathrm{FV}(u)$:

$$s[t/x][u/y] = s[u/y][t[u/y]/x]$$

Homework 4 (Trees in λ-calculus)

Encode a datatype of binary trees in lambda calculus. Specify the operations tip and node that construct trees, as well as isTip, left, right, and value. Each tip should carry a value, whereas each node should consist of two subtrees.

Show that the following holds:

$$\begin{aligned}
\mathsf{isTip}\ (\mathsf{tip}\ a) &\rightarrow_\beta^* \mathsf{true} \\
\mathsf{isTip}\ (\mathsf{node}\ x\ y) &\rightarrow_\beta^* \mathsf{false} \\
\mathsf{value}\ (\mathsf{tip}\ a) &\rightarrow_\beta^* a \\
\mathsf{left}\ (\mathsf{node}\ x\ y) &\rightarrow_\beta^* x \\
\mathsf{right}\ (\mathsf{node}\ x\ y) &\rightarrow_\beta^* y
\end{aligned}$$

Homework 5 (Alternative Encoding of Lists)

In this exercise, we consider an alternative encoding of lists. The list $[x, y, z]$, for instance, will now be encoded as: $\lambda cn.\ cx\,(cy\,(cz\,n))$ (speaking in terms of functional programming, each list now encodes its corresponding *fold*). As in the tutorial, define the functions nil, cons, hd, and null for this encoding and show that they satisfy the same conditions. You do not need to define tl.

Homework 6 (Multiplication)

Define multiplication as a closed λ-term using add but no other helper functions and demonstrate its correctness on an example.