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Lambda Calculus
Winter Term 2023/24
Exercise Sheet 4

Exercise 1 (β-reduction on de Bruijn Preserves Substitution)

We consider an alternative representation of λ-terms that is due to de Bruijn. In this representation, λ-terms are defined according to the following grammar:

$$d ::= i \in \mathbb{N}_0 \mid d_1\ d_2 \mid \lambda\ d$$

Remember that we defined substitution for de Bruijn terms as follows:

$$i \uparrow_l = \begin{cases} i, \text{ if } i < l \\ i+1, \text{ if } i \geq l \end{cases}$$
$$(d_1\ d_2) \uparrow_l = d_1 \uparrow_l\ \ d_2 \uparrow_l$$
$$(\lambda\ d) \uparrow_l = \lambda\ d \uparrow_{l+1}$$

$$i[t/j] = \begin{cases} i \text{ if } i < j \\ t \text{ if } i = j \\ i-1 \text{ if } i > j \end{cases}$$
$$(d_1\ d_2)[t/j] = (d_1[t/j])\ (d_2[t/j])$$
$$(\lambda\ d)[t/j] = \lambda\ (d[t \uparrow_0 /j+1])$$

For the β-reduction, we only need to modify the case of the substition. In particular, we define $(\lambda\ d)\ e \to_\beta d[e/0]$.

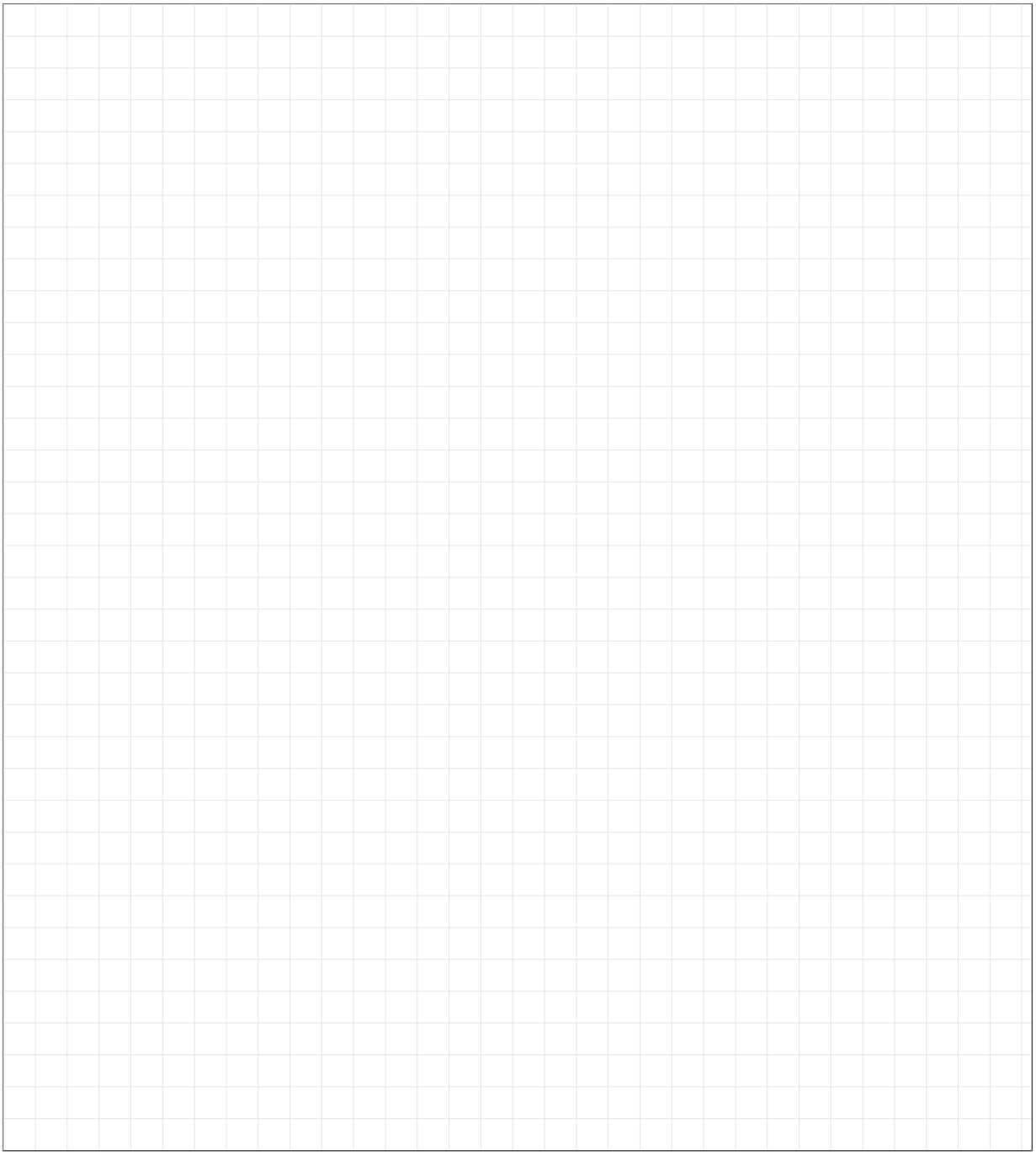
Prove that $s[u/i] \to_\beta s'[u/i]$ if $s \to_\beta s'$.

Exercise 2 (Strong Confluence)

A relation $\rightarrow$ is said to be *strongly confluent* iff:

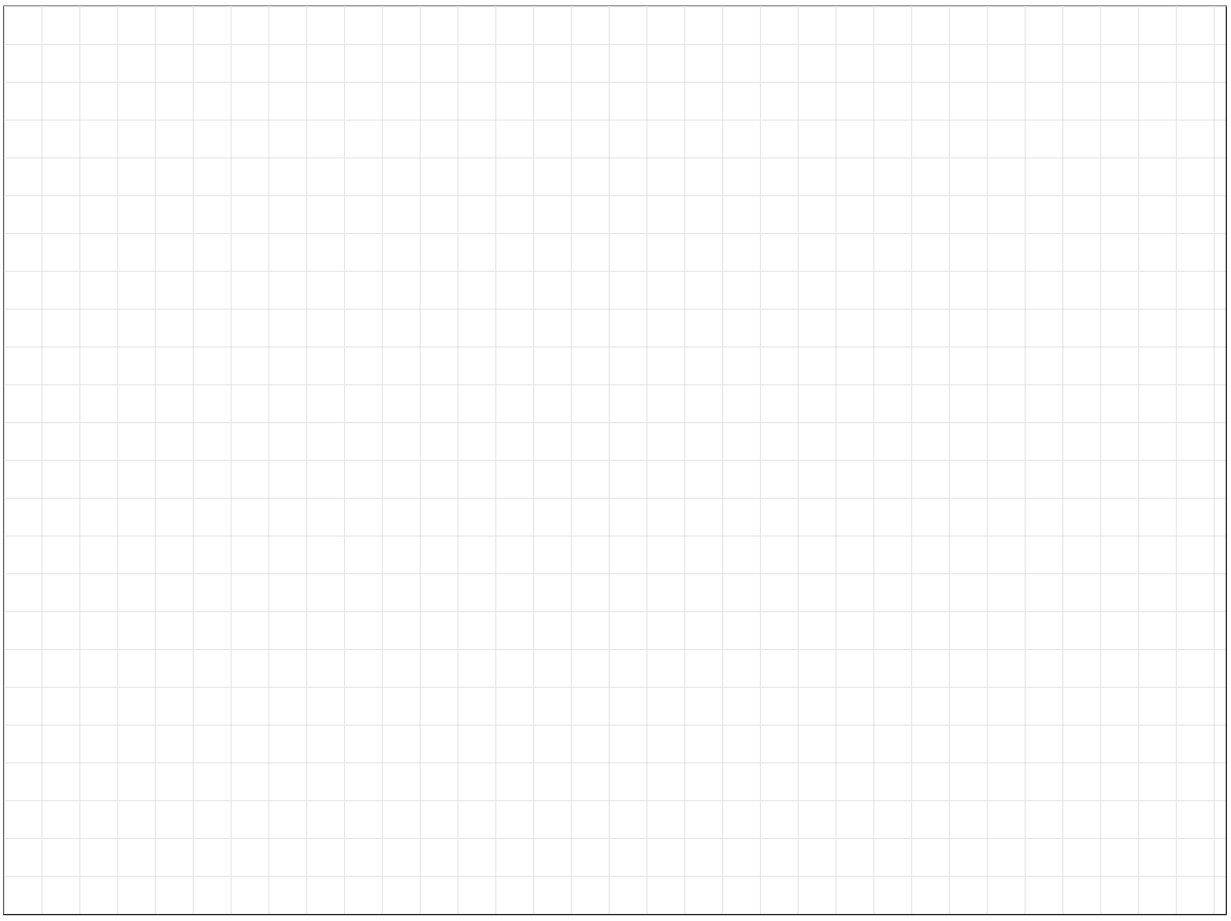
$$t_2 \leftarrow s \rightarrow t_1 \Longrightarrow \exists u.\ t_2 \rightarrow^{=} u\ {}^{*}\!\leftarrow t_1$$

Show that every *strongly confluent* relation is also *confluent*.



Exercise 3 (Diamond Property & Normal Forms)

Show that if $\to$ has the diamond property, every element is either in normal form or has no normal form.



Homework 4 (Semi-Confluence)

A relation $\rightarrow$ is said to be *semi-confluent* iff:

$$t_2 \ {}^*\!\leftarrow s \rightarrow t_1 \Longrightarrow \exists u.\ t_2 \rightarrow^* u \ {}^*\!\leftarrow t_1$$

Show that $\rightarrow$ is *semi-confluent* if and only if it is *confluent*.

Homework 5 (Weak Diamond Property)

Assume that $\rightarrow$ has the following weaker diamond property:

$$t_2 \leftarrow s \rightarrow t_1 \wedge t_1 \neq t_2 \Longrightarrow \exists u.\ t_2 \rightarrow u \leftarrow t_1.$$

a) Is it still the case that every element is either in normal form or has no normal form?

b) Show that if t has a normal form, then all its reductions to its normal form have the same length.