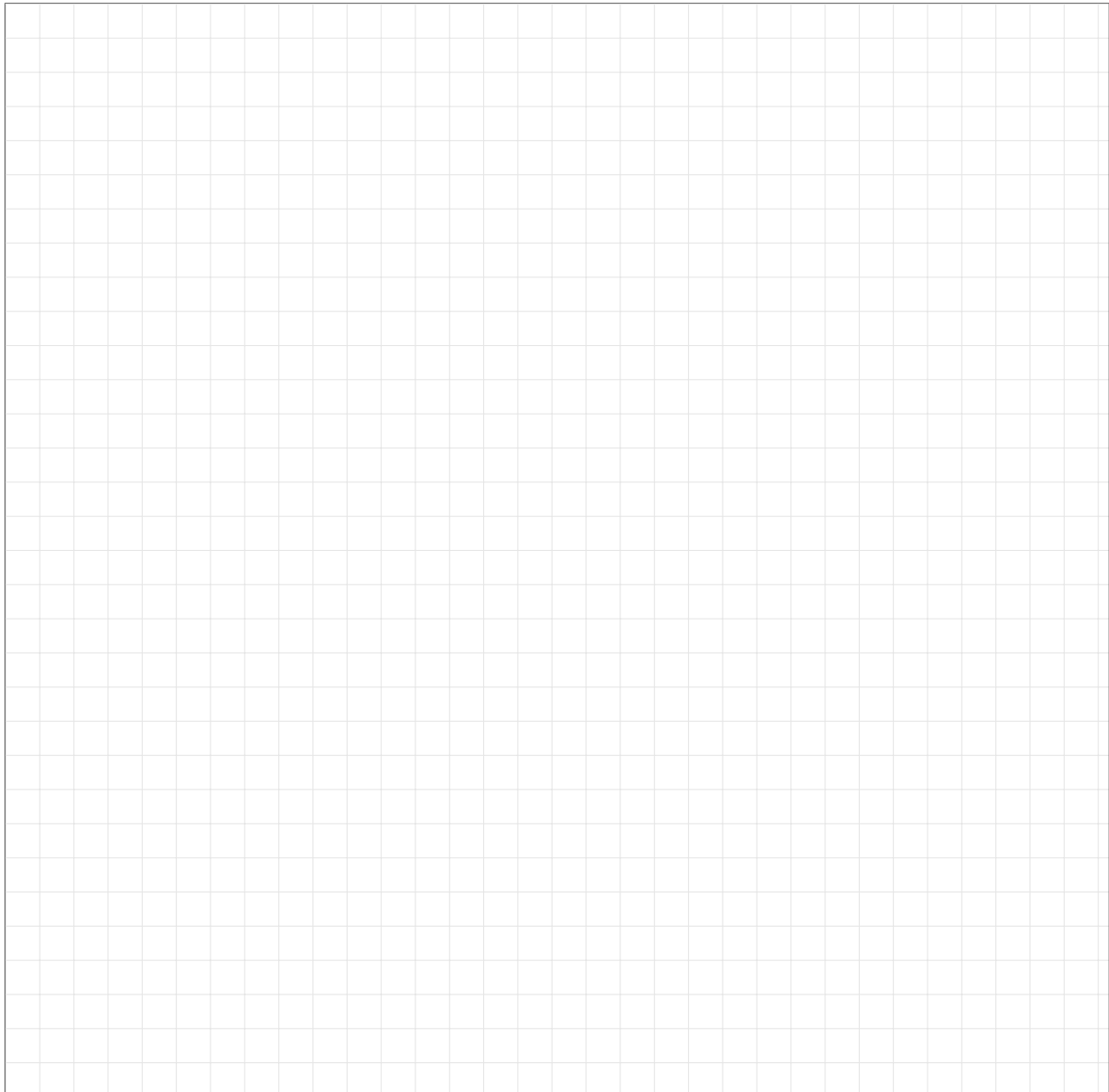


Exercise 1 (Confluence & Commutation)

Show: If \rightarrow_1 and \rightarrow_2 are confluent, and if \rightarrow_1^* and \rightarrow_2^* commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.



Exercise 2 (Confluence of β -Reduction with Takahashi functions)

In the lecture, we have shown the confluence of \rightarrow_β using the diamond property of parallel β -reduction. In this exercise, we develop an alternative proof based on what are sometimes called Takahashi functions. A function ρ is a Takahashi function with respect to a reduction relation $>$ if it holds that

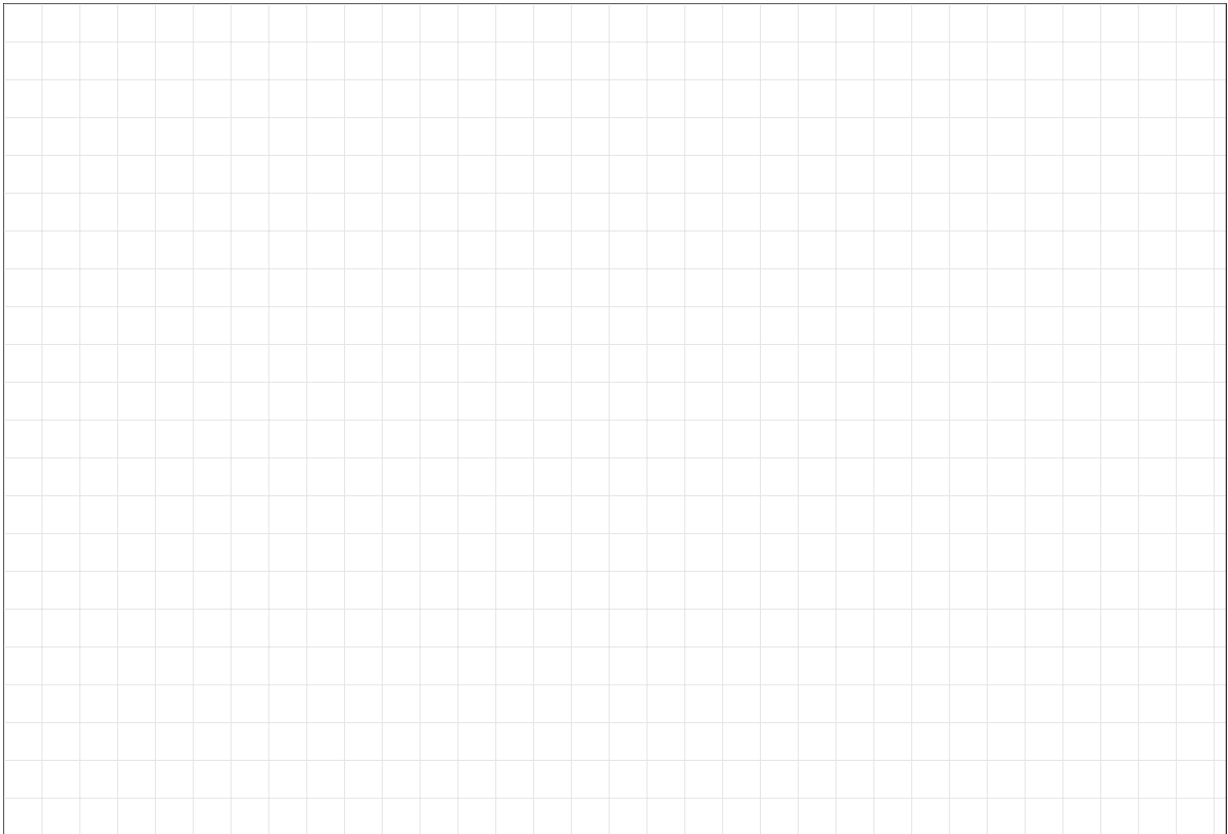
$$s > t \implies t > \rho(s).$$

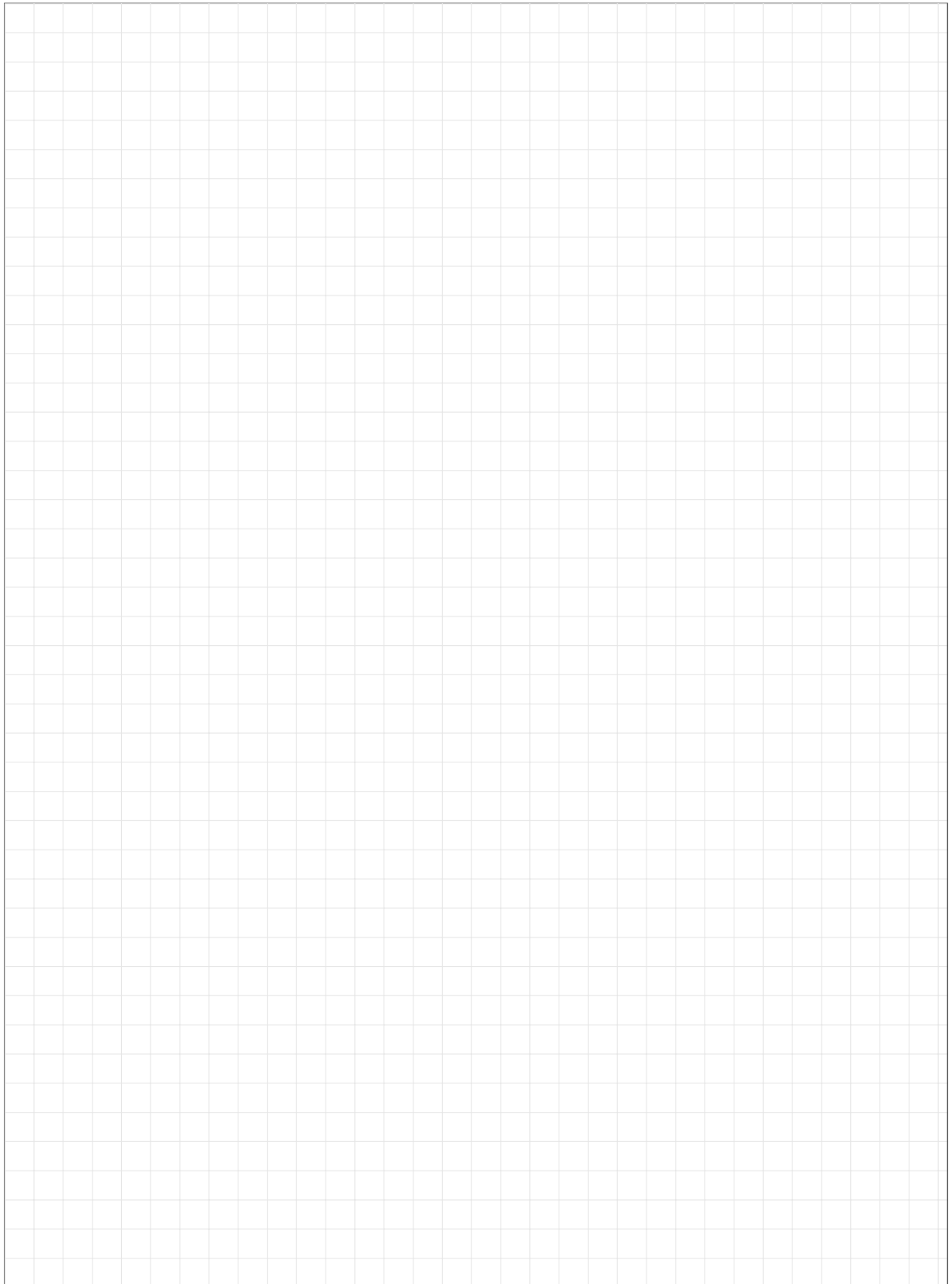
- a) Show that \rightarrow is confluent if it holds that $\rightarrow \subseteq > \subseteq \rightarrow^*$ and there exists a Takahashi function ρ for $>$.

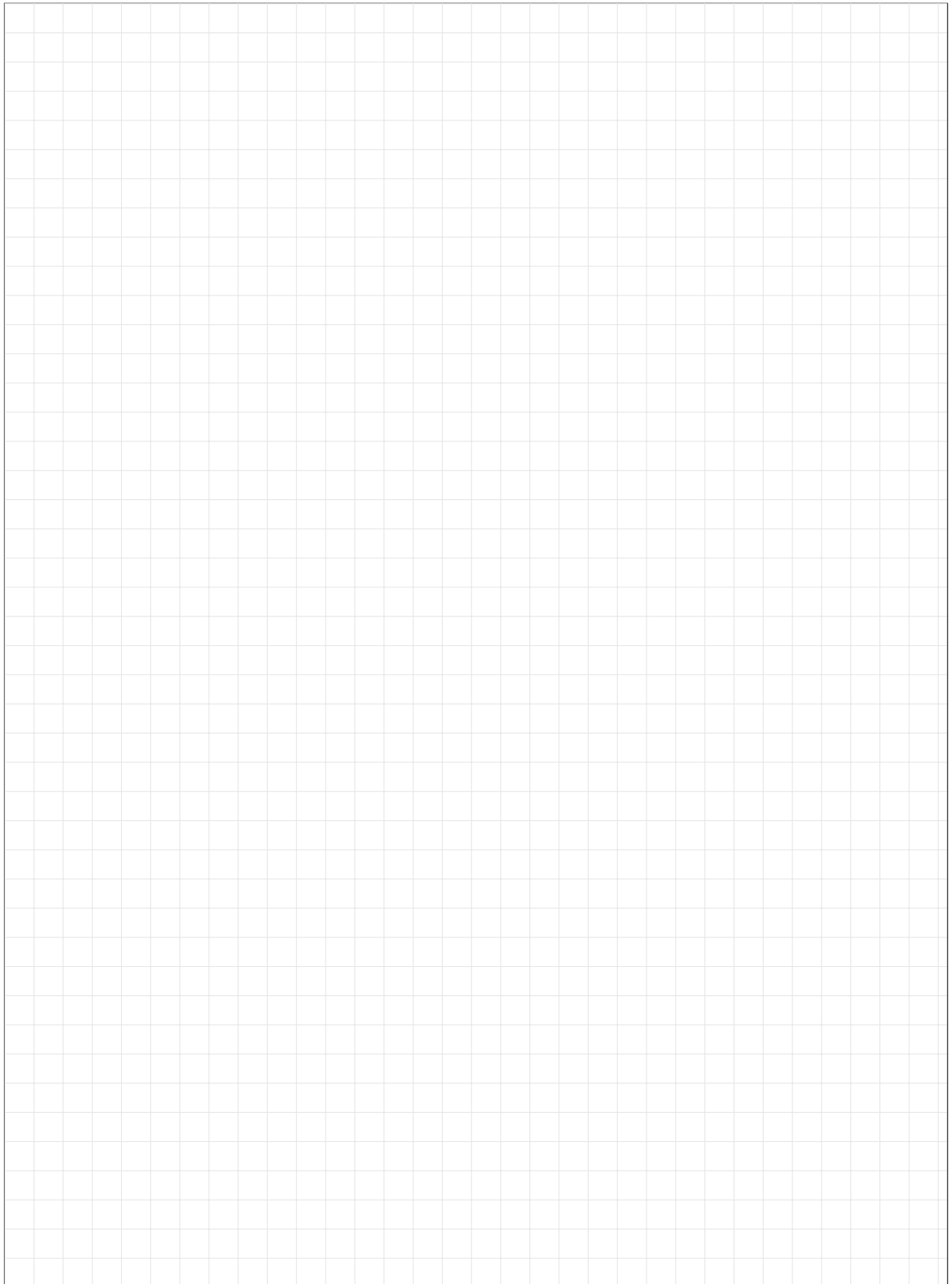
We define the operation $-^*$ on λ -terms inductively over the structure of terms:

$$\begin{aligned} x^* &= x \\ (\lambda x. t)^* &= \lambda x. t^* \\ (t_1 t_2)^* &= t_1^* t_2^* \quad \text{if } t_1 t_2 \text{ is not a } \beta\text{-redex.} \\ ((\lambda x. t_1) t_2)^* &= t_1^*[t_2^*/x] \end{aligned}$$

- b) Show that \rightarrow_β is confluent by proving that $-^*$ is a Takahashi function for the parallel and nested reduction $>$.



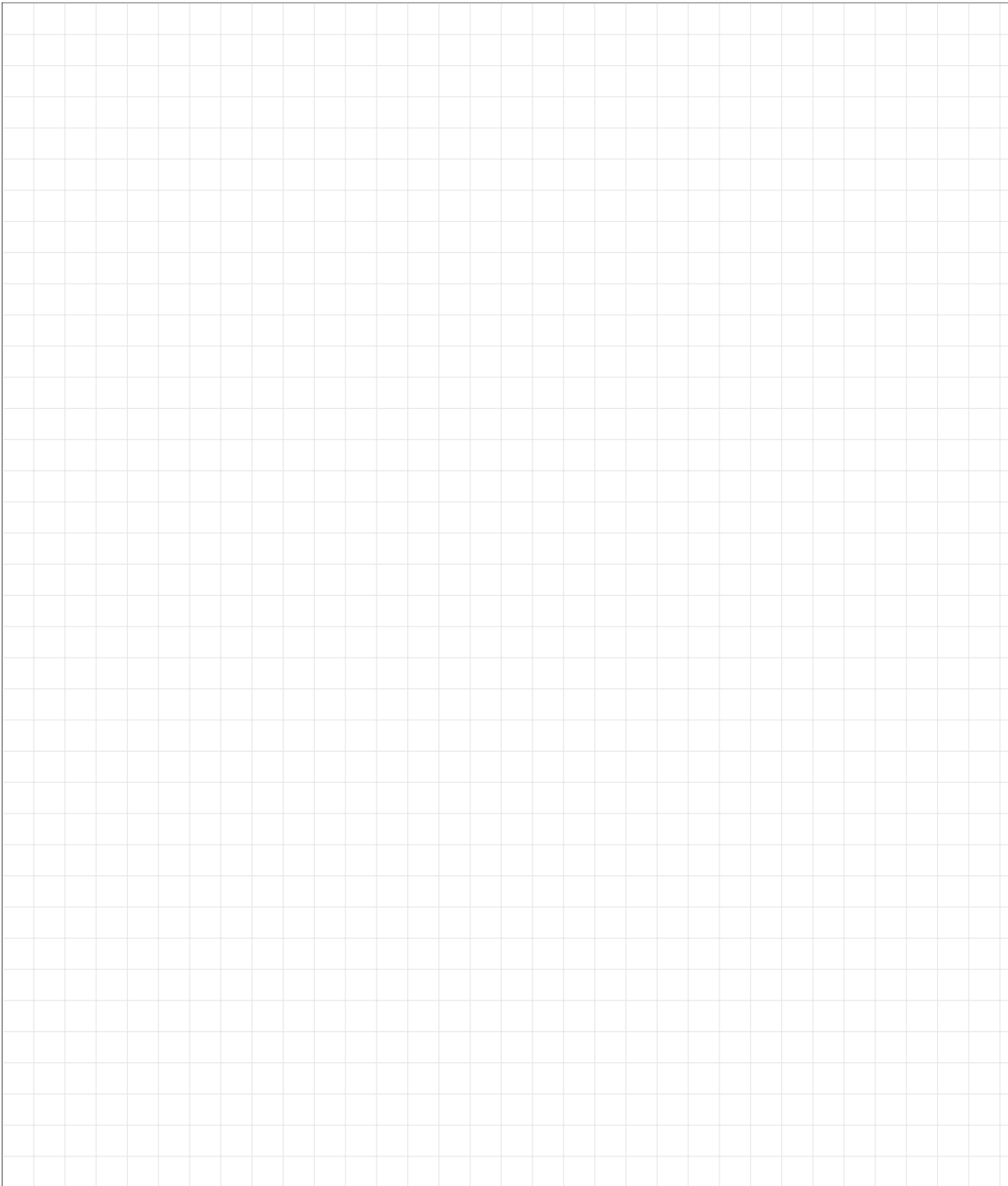




Exercise 3 (Parallel Beta Reduction)

Show:

$$s > t \implies s \rightarrow_{\beta}^* t$$



Homework 4 (Local Confluence of η -reduction)

Analogously to β -reduction, we define η -reduction inductively:

1. $x \notin \text{FV}(s) \implies (\lambda x. s x) \rightarrow_\eta s$
2. $s \rightarrow_\eta s' \implies s t \rightarrow_\eta s' t$
3. $s \rightarrow_\eta s' \implies t s \rightarrow_\eta t s'$
4. $s \rightarrow_\eta s' \implies (\lambda x. s) \rightarrow_\eta (\lambda x. s')$

The proof of local confluence of \rightarrow_η , i.e. it holds that there exists a u with $t_1 \rightarrow_\eta^* u \xleftarrow{*}_\eta t_2$ if we have $t_1 \xleftarrow{\eta} s \rightarrow_\eta t_2$, was very informal. Give a proper proof using this definition.

Homework 5 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \implies s[t/x] > s'[t'/x]$$

Homework 6 (A Takahashi function for combinatory logic)

Instead of the λ -calculus, we consider *combinatory logic* in this exercise whose syntax consists of variables, application, and the combinators **K** and **S**:

$$s, t ::= x \in \mathbb{N}_0 \mid s t \mid \mathbf{K} \mid \mathbf{S}.$$

We inductively define a reduction relation \rightarrow_w for this calculus with:

1. $\mathbf{K} s t \rightarrow_w s$
2. $\mathbf{S} s t u \rightarrow_w s u (t u)$
3. $s \rightarrow_w s' \implies s t \rightarrow_w s' t$
4. $t \rightarrow_w t' \implies s t \rightarrow_w s t'$

Use the strategy from the tutorial to prove that \rightarrow_w is confluent by defining a parallel and nested reduction relation $>_w$ for this calculus and a Takahashi function $-^*$ for $>_w$.