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Lambda Calculus
Winter Term 2023/24
Exercise Sheet 5

Exercise 1 (Confluence & Commutation)

Show: If $\rightarrow_1$ and $\rightarrow_2$ are confluent, and if $\rightarrow_1^*$ and $\rightarrow_2^*$ commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.

Exercise 2 (Confluence of β-Reduction with Takahashi functions)

In the lecture, we have shown the confluence of $\rightarrow_\beta$ using the diamond property of parallel β-reduction. In this exercise, we develop an alternative proof based on what are sometimes called Takahashi functions. A function ρ is a Takahashi function with respect to a reduction relation $>$ if it holds that

$$s > t \Longrightarrow t > \rho(s).$$

a) Show that $\rightarrow$ is confluent if it holds that $\rightarrow \subseteq > \subseteq \rightarrow^*$ and there exists a Takahashi function ρ for $>$.

We define the operation $-^*$ on λ-terms inductively over the structure of terms:

$$\begin{array}{rcll} x^* & = & x & \\ (\lambda x.\ t)^* & = & \lambda x.\ t^* & \\ (t_1\ t_2)^* & = & t_1^*\ t_2^* & \text{if } t_1\ t_2 \text{ is not a } \beta\text{-redex.} \\ ((\lambda x.\ t_1)\ t_2)^* & = & t_1^*[t_2^*/x] & \end{array}$$

b) Show that $\rightarrow_\beta$ is confluent by proving that $-^*$ is a Takahashi function for the parallel and nested reduction $>$.

Exercise 3 (Parallel Beta Reduction)

Show:

$$s > t \Longrightarrow s \rightarrow_\beta^* t$$

Homework 4 (Local Confluence of η-reduction)

Analogously to β-reduction, we define η-reduction inductively:

1. $x \notin \mathsf{FV}(s) \Longrightarrow (\lambda x.\ s\ x) \to_\eta s$
2. $s \to_\eta s' \Longrightarrow s\ t \to_\eta s'\ t$
3. $s \to_\eta s' \Longrightarrow t\ s \to_\eta t\ s'$
4. $s \to_\eta s' \Longrightarrow (\lambda x.\ s) \to_\eta (\lambda x.\ s')$

The proof of local confluence of $\to_\eta$, i.e. it holds that there exists a u with $t_1 \to_\eta^* u\ {}_\eta^*\!\leftarrow t_2$ if we have $t_1\ {}_\eta\!\leftarrow s \to_\eta t_2$, was very informal. Give a proper proof using this definition.

Homework 5 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \Longrightarrow s[t/x] > s'[t'/x]$$

Homework 6 (A Takahashi function for combinatory logic)

Instead of the λ-calculus, we consider *combinatory logic* in this exercise whose syntax consists of variables, application, and the combinators K and S:

$$s, t \quad ::= \quad x \in \mathbb{N}_0 \mid s\,t \mid \mathsf{K} \mid \mathsf{S}.$$

We inductively define a reduction relation $\to_{\mathsf{w}}$ for this calculus with:

1. $\mathsf{K}\,s\,t \to_{\mathsf{w}} s$
2. $\mathsf{S}\,s\,t\,u \to_{\mathsf{w}} s\,u\,(t\,u)$
3. $s \to_{\mathsf{w}} s' \Longrightarrow s\,t \to_{\mathsf{w}} s'\,t$
4. $t \to_{\mathsf{w}} t' \Longrightarrow s\,t \to_{\mathsf{w}} s\,t'$

Use the strategy from the tutorial to prove that $\to_{\mathsf{w}}$ is confluent by defining a parallel and nested reduction relation $>_{\mathsf{w}}$ for this calculus and a Takahashi function $-^*$ for $>_{\mathsf{w}}$.