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Lambda Calculus
Winter Term 2023/24
Exercise Sheet 9

Exercise 1 (Constraint Solving)

During type inference we generate type unification constraints between types of the form $\tau_1 \stackrel{?}{=} \kappa_1, \ldots, \tau_n \stackrel{?}{=} \kappa_n$. In order to solve these constraints, we want to find a substitution function σ such that $\sigma(\tau_i) = \sigma(\kappa_i)$ for $i \in \{1, \ldots, n\}$. For each of the following constraint systems, find such a substitution or justify that no such substitution exists. Additionally, give a λ-term that has the type $\sigma(\tau_0)$.

a) $\tau_0 \stackrel{?}{=} \tau_1 \to \tau_2,\ \tau_1 \stackrel{?}{=} \tau_2$

b) $\tau_0 \stackrel{?}{=} \tau_1 \to \tau_2,\ \tau_2 \stackrel{?}{=} \tau_3 \to \tau_4,\ \tau_4 \stackrel{?}{=} \tau_1$

c) $\tau_0 \stackrel{?}{=} \tau_1 \to \tau_2,\ \tau_2 \stackrel{?}{=} \tau_3 \to \tau_4,\ \tau_1 \stackrel{?}{=} \tau_5 \to \tau_4,\ \tau_1 \stackrel{?}{=} \tau_3 \to \tau_5$

d) $\tau_0 \stackrel{?}{=} \tau_1 \to \tau_2,\ \tau_1 \stackrel{?}{=} \tau_3 \to \tau_2,\ \tau_3 \stackrel{?}{=} \tau_1$

Exercise 2 (Type Inference in Haskell)

In this exercise, we will develop a type inference algorithm for the simply typed λ-calculus in Haskell. The general idea of the algorithm is to apply the type inference rules in a backward manner and to record equality constraints between types on the way. These constraints are then solved to obtain the result type. A template is available as `type_inference.hs`.

a) Take a look at the template provided on the website. We have provided definitions of terms and types in the simply typed λ-calculus, together with syntax sugar for input and printing. Moreover, you can find the type of substitutions and utility functions to work with substitutions, types and terms.

b) The first component of the algorithm is *unification* on types. Given a list of equality constraints between types of the form $u_1 \stackrel{?}{=} t_1, \ldots, u_n \stackrel{?}{=} t_n$, we want to produce a suitable substitution ϕ such that $\phi(u_i) = t_i$ for all $1 \leq i \leq n$ or report that the given constraints do not have a solution. Fill in the remaining cases of the function *solve* that achieves this functionality.

c) Now we want to apply the type inference rules and record the arising type constraints. Function *constraints* of type

$$\mathit{Term} \rightarrow \mathit{Type} \rightarrow \mathit{Env} \rightarrow \mathit{(Int,\ [(Type,\ Type)])} \rightarrow \mathit{Maybe\ (Int,\ [(Type,\ Type)])}$$

will achieve this functionality. Given a term t, a type τ, an enviroment Γ, and a pair (n, C), it will try to justify $\Gamma \vdash t \colon \tau$, adding the arising type constraints to C. The natural number n is used to keep track of the least variable index that is currently unused. This allows to easily generate fresh variable names. Complete the definition of *constraints*.

d) Define the function *infer* that infers the type of a term by combining *solve* and *constraints* and try it on a few examples.

Exercise 3 (Every Type is Applicative)

a) Show that every type is *substitutive*.

b) Show that every type is *applicative*.

Homework 4 (Types of Church Numerals)

a) Let τ be any type. Show that for the n-th Church numeral $\underline{\mathrm{n}}$, we have

$$[] \vdash \underline{\mathrm{n}} \colon (\tau \to \tau) \to \tau \to \tau.$$

b) Show that every term $t \in \mathrm{NF}$ with $[] \vdash t \colon (\iota \to \iota) \to \iota \to \iota$, t is either id or a church numeral. Here ι is any *elementary* type.

Homework 5 (Completeness of T)

In this exercise, you will show the converse of Lemma 3.2.2, i.e.

$$\Downarrow t \Longrightarrow t \in T$$

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a) Show that every λ-term has one of the following shapes:

- $x\, r_1 \ldots r_n$
- $\lambda x.\ r$
- $(\lambda x.\ r)\, s\, s_1\ \ldots\ s_n$

Note that this gives rise to an alternative inductive definition for λ-terms and to a corresponding rule induction on λ-terms.

b) $\Downarrow$ gives rise to a wellfounded induction principle. To show

$$\forall t.\ \Downarrow t \Longrightarrow P(t)\,,$$

it suffices to prove:

$$\forall t.\ (\forall t'.\ t \to_\beta t' \Longrightarrow P(t')) \Longrightarrow P(t)\,.$$

Use this to prove:

$$\Downarrow t \Longrightarrow t \in T$$

Hint: Use (a) for an inner induction on terms.