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Lambda Calculus
Winter Term 2023/24
Exercise Sheet 13

Exercise 1 (Church Numerals in System F)

Encode the natural numbers in System F with Church numerals. Use the construction for recursive types from the lecture.

Exercise 2 (Programming in System F)

System F allows us to define functions that go far beyond what was possible in the simply typed λ-calculus. In particular, we can also define some non-primitively recursive functions in System F. As a prominent example, consider the Ackermann function:

$$\begin{aligned} \mathsf{ack}\ 0\ n &= n + 1 \\ \mathsf{ack}\ (m+1)\ 0 &= \mathsf{ack}\ m\ 1 \\ \mathsf{ack}\ (m+1)\ (n+1) &= \mathsf{ack}\ m\ (\mathsf{ack}\ (m+1)\ n) \end{aligned}$$

Define the Ackermann function in System F based on the encoding of natural numbers from the last exercise. *Hint*: First define a function g such that $g\ f\ n = f^{n+1}\ \underline{1}$

Exercise 3 (Existential Quantification in System F)

System F can also be defined with additional existential types of the form $\exists\alpha.\ \tau$. To make use of these types, we add the following constructs to our term language

- $\mathsf{pack}\ \tau\ \mathsf{with}\ t\ \mathsf{as}\ \tau'$,
- $\mathsf{open}\ t\ \mathsf{as}\ \tau\ \mathsf{with}\ m\ \mathsf{in}\ t'$,

together with the reduction rule:

$$\mathsf{open}\ (\mathsf{pack}\ \tau\ \mathsf{with}\ t\ \mathsf{as}\ \exists\alpha.\ \tau')\ \mathsf{as}\ \alpha\ \mathsf{with}\ m\ \mathsf{in}\ t' \to t'[\tau/\alpha][t/m]$$

a) Specify the typing rules for $\exists$.

b) Show how $\exists$ can be used to specify an abstract module of sets that supports the empty set, insertion, and membership testing.

c) Show how to implement this module with lists.

d) How do these concepts relate to the SML (or OCaml) concepts of signatures, structures, and functors?

Homework 4 (Finger Exercises on Typing in System F)

a) Give a type τ such that

$$\vdash \lambda m\colon \mathsf{nat}.\ \ \lambda n\colon \mathsf{nat}.\ \lambda\, \alpha.\ (n\ (\alpha \to \alpha))\ (m\ \alpha)\colon \tau$$

is typeable in System F and prove the typing judgement. Recall that

$$\mathsf{nat} = \forall\, \alpha.\ (\alpha \to \alpha) \to \alpha \to \alpha\ .$$

b) Is there any typeable term t (in System F) such that if we remove all type annotations and type abstractions from t we get $(\lambda\, x.\ x\ x)\ (\lambda\, x.\ x\ x)$?

Homework 5 (Programming in System F)

Define (in System F) a function zero of type $\mathsf{nat} \to \mathsf{bool}$ that checks whether a given Church numeral is zero. Use the encoding that was introduced in the tutorial.

Homework 6 (Disjunction in System F)

Prove $\vee_{I_1}$ and $\vee_E$ from

$$A \vee B = \forall C.\ (A \to C) \to (B \to C) \to C$$

in System F. Use pure logic without lambda-terms.

Homework 7 (Progress and Preservation)

We have proved the properties of *progress* (see Exercise 7.1) and *preservation* (see Homework 7.4) for the simply typed λ-calculus. Extend our previous proofs to show that these properties also hold for System F.