



Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
- This number is printed both next to the code and to the signature field in the attendance check list.

Lambda Calculus

Exam: IN2358 / Retake **Date:** Friday 5th April, 2024
Examiner: Prof. Tobias Nipkow **Time:** 17:00 – 18:30

	P 1	P 2	P 3	P 4	P 5
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Working instructions

- This exam consists of **12 pages** with a total of **5 problems**.
Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 30 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
 - one **DIN A4 sheet** with **hand-written** notes on both sides
 - one **analog dictionary** English ↔ native language
- Subproblems marked by * can be solved without results of previous subproblems.
- **Answers are only accepted if the solution approach is documented.** Give a reason for each answer unless explicitly stated otherwise in the respective subproblem.
- Do not write with red or green colors nor use pencils.
- Physically turn off all electronic devices, put them into your bag and close the bag.

Left room from _____ to _____ / Early submission at _____

Problem 1 Programming with λ -terms (12 credits)

Recall the fold encoding from the exercise sheets where a list $[x, y, z]$ is represented as $\lambda c n. c x (c y (c z n))$. Accordingly, the empty list is defined as $\text{nil} := \lambda c n. n$ and we can prepend an element to a list with $\text{cons } x l := \lambda c n. c x (l c n)$.

0 a)* Define a λ -term **snoc** that appends one element, i.e. it should hold that

- 1 • $\text{snoc } x \text{ nil} =_{\beta} \text{cons } x \text{ nil}$
- 2 • $\text{snoc } x (\text{cons } h t) =_{\beta} \text{cons } h (\text{snoc } x t)$

3
4 $\text{snoc} :=$

0 b)* Without using a fixed point combinator, define a λ -term **reverse** that reverses a list, i.e. it should hold that

- 1 • $\text{reverse } \text{nil} =_{\beta} \text{nil}$
- 2 • $\text{reverse } (\text{cons } h t) =_{\beta} \text{snoc } h (\text{reverse } t)$

3
4 $\text{reverse} :=$

0 c)* Implement **reverse** again using the fixed point combinator **fix**. Your implementation must not rely on the underlying encoding of lists but you may use the functions **nil**, **cons**, **null**, **head**, **tail** with the usual semantics. Additionally, you may use **snoc**. Remember that **null** returns a boolean, i.e. $\text{null } \text{nil } x y \rightarrow_{\beta}^* x$ and $\text{null } (\text{cons } h t) x y \rightarrow_{\beta}^* y$.

1
2
3
4 $\text{reverse} :=$

Problem 2 Rewriting (8 credits)

Let $\rightarrow \subseteq A \times A$ be a relation. Below, all variables are implicitly assumed to be elements of A .

The set S of **direct successors** of x is defined as $S(x) = \{y \mid x \rightarrow y\}$.

We call \rightarrow

finitely branching if every x has only finitely many direct successors,
i.e. $S(x)$ is finite.

\mathbb{N} -terminating if there is a function $t : A \rightarrow \mathbb{N}$ such that
 $x \rightarrow y \implies t(x) > t(y)$.

An element x is called **bounded**, or more explicitly **k -bounded**, if the length of all reductions starting from x is bounded: there is a $k \in \mathbb{N}$ such that for every y it holds that $x \rightarrow^n y \implies n \leq k$.

Assume that \rightarrow is finitely branching and \mathbb{N} -terminating. Prove that all x are bounded.

Hint: Proof by induction.

The proof must be given in the standard verbal style.



Problem 3 Quiz (3 credits)



a)* We define the set of WNF inductively with

$$\frac{t_1, \dots, t_n \in \text{WNF}}{x \ t_1 \ \dots \ t_n \in \text{WNF}}$$
$$\frac{}{(\lambda x. t) \in \text{WNF}}$$

Give a closed term t such that $t \in \text{WNF}$ but $t \notin \text{NF}$. Justify your choice of t !



b)* Give a type τ such that only finitely many simply typed λ -terms t have that type, i.e. it holds that $\vdash t : \tau$.



c)* Let f be a λ -term with type $\mathbf{bool} \rightarrow \mathbf{bool}$ in System F, i.e. it holds that $\vdash f : \mathbf{bool} \rightarrow \mathbf{bool}$. Is it decidable whether $f \ x \rightarrow_{\beta}^* \mathbf{true}$ for every x in $\beta\eta$ -normal form with type \mathbf{bool} ? Remember that there are exactly two terms with type \mathbf{bool} that are in $\beta\eta$ -normal form, namely \mathbf{true} and \mathbf{false} .

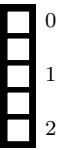
Problem 4 Typing with let-polymorphism (5 credits)

Consider simply typed lambda calculus extended with `let` and consider the following the typing problem

$$a : A \vdash \text{let } x = \lambda z. z a \text{ in } \lambda y. x (x y) : ?\tau$$

where A is a type variable.

a)* Find a most general type schema σ with $a : A \vdash \lambda z. z a : \sigma$. You may, but you do not need to draw a type derivation tree.



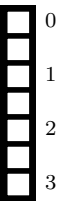
b) Draw the type derivation tree for

$$a : A, x : \sigma \vdash \lambda y. x (x y) : ?\tau$$

Of course, with the correct type for $?\tau$.

Use only the introduction and elimination rules for \rightarrow and \forall and the standard assumption rule

$$\Gamma \vdash x : \tau \quad \text{where} \quad \Gamma(x) = \tau.$$



Problem 5 Intuitionistic Logic (2 credits)

In this exercise, we consider intuitionistic logic with just the rules $\wedge I$, $\wedge E_1$, $\wedge E_2$, $\neg I$, and $\neg E$, i.e.

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \neg I$$

$$\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash \perp} \neg E$$

In the following, explicitly annotate the rule that was used for each inference step.

- 0 a)* Prove that an assumption A can be weakened to $A \wedge B$ in intuitionistic logic, i.e. show that $\Gamma, A \vdash C$
1 implies $\Gamma, A \wedge B \vdash C$. Use induction on the derivation of $\Gamma, A \vdash C$. You only need to consider the cases where $\Gamma, A \vdash C$ was proved by assumption or $\wedge I$.

b)* In order to represent implication, we view $A \rightarrow B$ as an abbreviation for $\neg(A \wedge \neg B)$. Prove the implication introduction rule, i.e. show that $\Gamma, A \vdash B$ implies $\Gamma \vdash A \rightarrow B$.

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Additional space for solutions—clearly mark the (sub)problem your answers are related to and strike out invalid solutions.

