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Lambda Calculus
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Solutions to Exercise Sheet 5

Exercise 1 (Confluence & Commutation)

Show: If $\rightarrow_1$ and $\rightarrow_2$ are confluent, and if $\rightarrow_1^*$ and $\rightarrow_2^*$ commute, then $\rightarrow_{12} := \rightarrow_1 \cup \rightarrow_2$ is also confluent.

Solution

Lemma A.3.2 from the lecture. The key idea is to consider $\rightarrow_1^* \circ \rightarrow_2^*$ as $\rightarrow_{12}^*$ unfolds into iterations of this relation, i.e. $(\rightarrow_1^* \circ \rightarrow_2^*)^* = \rightarrow_{12}^*$. More precisely:

$$\rightarrow_{12} \subseteq \rightarrow_1^* \circ \rightarrow_2^* \subseteq \rightarrow_{12}^* \qquad (*)$$

The relation $\rightarrow_1^* \circ \rightarrow_2^*$ has the diamond property:

$$\begin{tikzcd}
\bullet \arrow[r, "*", "1"'] \arrow[d, "1", "*"'] \arrow[dr, phantom, "\text{confluence of } \rightarrow_1"] & \bullet \arrow[r, "*", "2"'] \arrow[d, "1", "*"'] \arrow[dr, phantom, "\text{commutativitiy}"] & \bullet \arrow[d, "1", "*"'] \\
\bullet \arrow[r, "*", "1"'] \arrow[d, "2", "*"'] \arrow[dr, phantom, "\text{commutativitiy}"] & \bullet \arrow[r, "*", "2"'] \arrow[d, "2", "*"'] \arrow[dr, phantom, "\text{confluence of } \rightarrow_2"] & \bullet \arrow[d, "2", "*"'] \\
\bullet \arrow[r, "*", "1"'] & \bullet \arrow[r, "*", "2"'] & \bullet
\end{tikzcd}$$

With (*) and Lemma A.2.5 it immediately follows that $\rightarrow_{12}$ is confluent.

Exercise 2 (Confluence of β-Reduction with Takahashi functions)

In the lecture, we have shown the confluence of $\rightarrow_\beta$ using the diamond property of parallel β-reduction. In this exercise, we develop an alternative proof based on what are sometimes called Takahashi functions. A function ρ is a Takahashi function with respect to a reduction relation $>$ if it holds that

$$s > t \Longrightarrow t > \rho(s).$$

a) Show that $\rightarrow$ is confluent if it holds that $\rightarrow \subseteq > \subseteq \rightarrow^*$ and there exists a Takahashi function ρ for $>$.

We define the operation $-^*$ on λ-terms inductively over the structure of terms:

$$\begin{aligned} x^* &= x \\ (\lambda x.\ t)^* &= \lambda x.\ t^* \\ (t_1\ t_2)^* &= t_1^*\ t_2^* \qquad \text{if } t_1\ t_2 \text{ is not a } \beta\text{-redex.} \\ ((\lambda x.\ t_1)\ t_2)^* &= t_1^*[t_2^*/x] \end{aligned}$$

b) Show that $\to_\beta$ is confluent by proving that $-^*$ is a Takahashi function for the parallel and nested reduction $>$.

Solution

a) Since ρ has the Takahashi-property, it follows that $>$ has the diamond property: Let $s > t_1$ and $s > t_2$. Then, we have both $t_1 > \rho(s)$ and $t_2 > \rho(s)$. Like in the lecture, we have that

$$\to \subseteq > \subseteq \to^* \Longrightarrow \to^* \subseteq >^* \subseteq \to^* \Longleftrightarrow >^* = \to^*$$

and since $>$ has the diamond property it follows that $\to$ is confluent.

b) We prove that $-^*$ has the Takahashi property, i.e.

$$s > t \quad \Longrightarrow \quad t > s^*,$$

by $-^*$ computation induction on s for arbitrary t. In the following, we will skip instances of $s > s$, as we can easily prove $s > s^*$ by another structural induction on s.

1st case: $s = x$.

Assume $x > t$. Hence $t = x$. Thus $t = x > x = x^* = s^*$.

2nd case: $s = \lambda x.\ s_1$.

Assume $\lambda x.\ s_1 > t$. Furthermore, we get the induction hypothesis $s_1 > t \Longrightarrow t > s_1^*$ for arbitrary t. By case analysis on the derivation of $s > t$ we obtain $t = \lambda x.\ t_1$ for an appropriate t_1 and we have $s_1 > t_1$. This implies $t_1 > s_1^*$ considering the induction hypothesis. Thus we get $t = \lambda x.\ t_1 > \lambda x.\ s_1^* = (\lambda x.\ s_1)^* = s^*$.

3rd case: $s = s_1\ s_2$, but s is not a β-redex.

Assume $s_1\ s_2 > t$. As induction hypotheses we obtain $s_i > t \Longrightarrow t > s_i^*$ for arbitrary t and $i \in \{1, 2\}$. Then $t = t_1\ t_2$ with $s_1 > t_1$ and $s_2 > t_2$ (again this follows by case analysis using the inductive definition of $>$). We get the induction hypotheses $t_1 > s_1^*$ and $t_2 > s_2^*$. By the definition of $>$ this implies $t = t_1\ t_2 > s_1^*\ s_2^* = (s_1\ s_2)^* = s^*$.

4th case: $s = (\lambda x.\ s_1)\ s_2$.

Assume $(\lambda x.\ s_1)\ s_2 > t$. Then there is t_1, t_2 with $s_1 > t_1$ and $s_2 > t_2$ and we have $t = (\lambda x.\ t_1)\ t_2$ or $t = t_1[t_2/x]$, depending on the rule used to derive $s > t$. Note that the first case actually requires a nested case distinction: first we consider the case that $t = t'_1\ t_2$ by the third rule in the definition of $>$. This requires that $(\lambda x.\ s_1) > t'_1$. Hence, $t'_1 = (\lambda x.\ t_1)$ and $s_1 > t_1$ follows by another case distinction on $(\lambda x.\ s_1) > t'_1$.

Case 4.1 $t = (\lambda x.\ t_1)\ t_2$.

By the induction hypotheses $t_1 > s_1^*$ and $t_2 > s_2^*$, we get $t = (\lambda x.\ t_1)\ t_2 > s_1^*[s_2^*/x] = ((\lambda x.\ s_1)\ s_2)^* = s^*$.

Case 4.2 $t = t_1[t_2/x]$.

By the induction hypotheses $t_1 > s_1^*$ and $t_2 > s_2^*$, and the substitution propery of $>$ (Lemma 1.2.13), we get $t = t_1[t_2/x] > s_1^*[s_2^*/x] = ((\lambda x.\ s_1)\ s_2)^* = s^*$.

A different proof is by structural induction on s for arbitrary t.

Case $s = x$.

Assume $x > t$. Hence $t = x$ by case analysis on the $x > t$. Thus $t = x > x = x^* = s^*$.

Case $s = \lambda x.\ s_1$.

Assume $\lambda x.\ s_1 > t$. Furthermore, we get the induction hypothesis $s_1 > t \Longrightarrow t > s_1^*$ for arbitrary t. By case analysis on the derivation of $s > t$ we obtain $t = \lambda x.\ t_1$ for an appropriate t_1 and we have $s_1 > t_1$. This, together with the induction hypothesis, implies $t_1 > s_1^*$. Thus, we get $t = \lambda x.\ t_1 > \lambda x.\ s_1^* = (\lambda x.\ s_1)^* = s^*$.

Case $s = s_1\ s_2$.

Assume $s_1\ s_2 > t$. As induction hypotheses we obtain $s_i > t \Longrightarrow t > s_i^*$ for arbitrary t and $i \in \{1, 2\}$. Case analysis on the derivation of $s_1\ s_2 > t$ yields two cases.

- We have $t = t_1\ t_2$ and $s_1 > t_1$ as well as $s_2 > t_2$, or
- $s_1 = (\lambda x.\ s_3)$ and $t = t_1[t_2/x]$ as well as $s_3 > t_1$ and $s_2 > t_2$.

Case $t = t_1\ t_2$.

From the induction hypotheses we have $t_1 > s_1^*$ and $t_2 > s_2^*$. This implies that $t = t_1\ t_2 > s_1^*\ s_2^* = (s_1\ s_2)^* = s^*$

Case $t = t_1[t_2/x]$.

We can only discharge the assumption of the second induction hypothesis which gives us $t_2 > s_2^*$. To obtain $t_1 > s_3^*$, we actually need an induction on the size of the term s — or a strong induction on the structure of s — such that the I.H. holds for all proper subterms of s (in particular s_3). Together with the substitution propery of $>$ (Lemma 1.2.13), we get $t = t_1[t_2/x] > s_3^*[s_2^*/x] = ((\lambda x.\ s_3)\ s_2)^* = s^*$.

Exercise 3 (Parallel Beta Reduction)

Show:

$$s > t \Longrightarrow s \rightarrow_\beta^* t$$

Solution

Proof by rule induction on $>$.

Case $s > s$**:** We have that $s = t$ and $s \rightarrow_\beta^* s$.

Case $\lambda x.\ s > \lambda x.\ s'$**:** As an induction hypothesis we get $s \rightarrow_\beta^* s'$. By Lemma 1.2.3 we obtain our goal $\lambda x.\ s \rightarrow_\beta^* \lambda x.\ s'$.

Case $s\ t > s'\ t'$**:** As induction hypotheses we get $s \rightarrow_\beta^* s'$ and $t \rightarrow_\beta^* t'$. With Lemma 1.2.3 we get $s\ t \rightarrow_\beta^* s'\ t \rightarrow_\beta^* s'\ t'$ and therefore $s\ t \rightarrow_\beta^* s'\ t'$.

Case $(\lambda x.\ s)\ t > s'[t'/x]$**:** As induction hypotheses we get $s \rightarrow_\beta^* s'$ and $t \rightarrow_\beta^* t'$. With Lemma 1.2.3 it holds that $(\lambda x.\ s)\ t \rightarrow_\beta^* (\lambda x.\ s')\ t \rightarrow_\beta^* (\lambda x.\ s')\ t' \rightarrow_\beta s'[t'/x]$ and thus $(\lambda x.\ s)\ t \rightarrow_\beta^* (\lambda x.\ s')\ t'$.

Homework 4 (Local Confluence of η-reduction)

Analogously to β-reduction, we define η-reduction inductively:

1. $x \notin \mathsf{FV}(s) \Longrightarrow (\lambda x.\ s\ x) \to_\eta s$
2. $s \to_\eta s' \Longrightarrow s\ t \to_\eta s'\ t$
3. $s \to_\eta s' \Longrightarrow t\ s \to_\eta t\ s'$
4. $s \to_\eta s' \Longrightarrow (\lambda x.\ s) \to_\eta (\lambda x.\ s')$

The proof of local confluence of $\to_\eta$, i.e. it holds that there exists a u with $t_1 \to_\eta^* u \ {}_\eta^*\!\leftarrow t_2$ if we have $t_1 \ {}_\eta\!\leftarrow s \to_\eta t_2$, was very informal. Give a proper proof using this definition.

Homework 5 (Parallel Beta Reduction & Substitution)

Show:

$$s > s' \wedge t > t' \Longrightarrow s[t/x] > s'[t'/x]$$

Homework 6 (A Takahashi function for combinatory logic)

Instead of the λ-calculus, we consider *combinatory logic* in this exercise whose syntax consists of variables, application, and the combinators K and S:

$$s, t \quad ::= \quad x \in \mathbb{N}_0 \mid s\,t \mid \mathsf{K} \mid \mathsf{S}.$$

We inductively define a reduction relation $\to_{\mathsf{w}}$ for this calculus with:

1. $\mathsf{K}\,s\,t \to_{\mathsf{w}} s$
2. $\mathsf{S}\,s\,t\,u \to_{\mathsf{w}} s\,u\,(t\,u)$
3. $s \to_{\mathsf{w}} s' \Longrightarrow s\,t \to_{\mathsf{w}} s'\,t$
4. $t \to_{\mathsf{w}} t' \Longrightarrow s\,t \to_{\mathsf{w}} s\,t'$

Use the strategy from the tutorial to prove that $\to_{\mathsf{w}}$ is confluent by defining a parallel and nested reduction relation $>_{\mathsf{w}}$ for this calculus and a Takahashi function $-^*$ for $>_{\mathsf{w}}$.