Technische Universität München
Institut für Informatik
Prof. Tobias Nipkow, Ph.D.
Lukas Stevens

Lambda Calculus
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Solutions to Exercise Sheet 6

Exercise 1 (Normal Forms)

Recall the inductive definition of the set NF of *normal forms*:

$$\frac{t \in \mathrm{NF}}{\lambda x.\ t \in \mathrm{NF}}$$

$$\frac{n \geq 0 \qquad t_1 \in \mathrm{NF} \qquad t_2 \in \mathrm{NF} \qquad \dots \qquad t_n \in \mathrm{NF}}{x\ t_1\ t_2\ \dots\ t_n \in \mathrm{NF}}$$

Show that this set precisely captures all normal forms, i.e.:

$$t \in \mathrm{NF} \Leftrightarrow \nexists t'.\ t \to_\beta t'$$

Solution

Direction $\Longrightarrow$: Proof by induction on the derivation of $t \in \mathrm{NF}$.

Case $t = \lambda x.\ s \in \mathrm{NF}$
We obtain the induction hypothesis $s \in \mathrm{NF} \Longrightarrow \nexists s'.\ s \to_\beta s'$. Suppose there exists an s' such that $\lambda x.\ s \to_\beta \lambda x.\ s'$. By rule inversion, it must hold that $s \to_\beta s'$ which is a contradiction to induction hypothesis where we discharge the assumption $s \in \mathrm{NF}$ by rule inversion on $t \in \mathrm{NF}$.

Case $t = x\ t_1\ \dots\ t_n \in \mathrm{NF}$
By rule inversion on $t \in \mathrm{NF}$ we get $t_i \in \mathrm{NF}$ and therefore $\nexists t'.\ t_i \to_\beta t'$ by the IH for $1 \leq i \leq n$. We show this case by another induction on n. The case $n = 0$ is trivial. Now assume $\nexists t'.\ x\ t_1\ t_2\ \dots\ t_n \to_\beta t'$ as the induction hypothesis, and $x\ t_1\ t_2\ \dots\ t_n\ t_{n+1} \to_\beta t'$ for the sake of contradiction. By analysing the derivation of the latter, we can only conlude $\exists t'.\ t_{n+1} \to_\beta t'$, which manifests a contradiction.

Direction $\Longleftarrow$: We show $t \notin \mathrm{NF} \Longrightarrow \exists t'.\ t \to_\beta t'$ by structural induction on t.

Case $t = x$
Then $t \in \mathrm{NF}$ which is a contradiction to the assumption $t \notin \mathrm{NF}$.

Case $t = \lambda x.\ s$
We assume that $t = \lambda x.\ s \notin \mathrm{NF}$. Thus, it must hold that $s \notin \mathrm{NF}$ by rule inversion, which together with the IH lets us obtain a s' where $s \to_\beta s'$. Hence, we conclude that $\lambda x.\ s \to_\beta \lambda x.\ s'$.

Case $t = t_1\ t_2$

We assume $t_1 \notin \mathrm{NF} \implies \exists t'.\ t_1 \to_\beta t'$ and $t_2 \notin \mathrm{NF} \implies \exists t'.\ t_2 \to_\beta t'$ as the induction hypothesis, and $t \notin \mathrm{NF}$.

The cases where $t_1 \notin \mathrm{NF}$ or $t_2 \notin \mathrm{NF}$ are immediate by the induction hypotheses together with the definition of $\to_\beta$.

Consider the case $t_1, t_2 \in \mathrm{NF}$. We analyze the derivation of $t_1 \in \mathrm{NF}$.

Case $t_1 = x\ s_1\ \dots\ s_n$ **where** $s_1, \dots, s_n \in \mathrm{NF}$

It follows that $t = t_1\ t_2 = x\ s_1\ \dots\ s_n\ t_2 \in \mathrm{NF}$ since $t_2 \in \mathrm{NF}$. But this is a contradiction to the assumption $t_1\ t_2 \notin \mathrm{NF}$.

Case $t_1 = (\lambda x.\ s)$ **where** $s \in \mathrm{NF}$

Then it holds that $t = t_1\ t_2 = (\lambda x.\ s)\ t_2 \to_\beta s[t_2/x]$.

Exercise 2 (Weakly Normalising Terms)

Inductively define the set of weakly normalising terms WN, i.e. the set of terms that have a β-normal form. In particular it should hold that

$$s \in \mathrm{WN} \iff \exists t.\ s \Rightarrow_n t.$$

Similarly, define the set of strongly normalising terms SN where a term s is strongly normalising if there is no infinite sequence $\{t_i \mid i \in \mathbb{N}\}$ with $s \to_\beta^* t_0$ and $t_i \to_\beta t_{i+1}$ for $i \in \mathbb{N}$.

Give a term t that is weakly but not strongly normalising.

Solution

We define WN as follows:

1. $t_1, \dots, t_m \in \mathrm{WN} \implies x\ t_1\ \dots\ t_m \in \mathrm{WN}$
2. $t \in \mathrm{WN} \implies \lambda x.\ t \in \mathrm{WN}$
3. $s[t/x]\ t_1\ \dots\ t_m \in \mathrm{WN} \implies (\lambda x.\ s)\ t\ t_1\ \dots\ t_m \in \mathrm{WN}$

The definition of SN is identical with exception of the last rule where we also demand that $t \in \mathrm{SN}$.

Let $\omega := (\lambda x.\ x\ x)$. Then, it holds that $(\lambda x.\ y)\ (\omega\ \omega) \in \mathrm{WN} \setminus \mathrm{SN}$.

Homework 3 (Normal Forms & Big Step)

Show:

$$t \in \mathrm{NF} \land t \Rightarrow_n u \Longrightarrow u = t$$

Homework 4 (Characterisation of WN)

Prove the following characterisation of WN:

$$t \in \mathrm{WN} \iff \exists t'.\, t \to_\beta^* t' \land t' \in \mathrm{NF}$$

You may use the fact that

$$s \in \mathrm{WN} \iff \exists t.\, s \Rightarrow_n t$$

and all theorems up to Theorem 1.5.8 from the lecture.