

Exercise 1 (Intuitionistic Proof Search)

The algorithm in Theorem 4.1.4 can be streamlined as follows:

- When trying to prove $\Gamma \vdash A \rightarrow B$, it suffices to try (\rightarrow Intro). Explain why.
- The attempt to prove $\Gamma \vdash A$ by assumption can be dropped: it is subsumed by the alternative using Lemma 4.1.2. However, the proof obtained can be different. Explain the difference and why the outright proof by assumption is subsumed.
- How would the [Haskell code](#) from the last tutorial need to be adapted to account for these improvements?

Solution

In the following we will denote the by (\rightarrow Elim) the more general rule described in lemma 4.1.2.

- Suppose we prove $\Gamma \vdash A \rightarrow B$ by an application of (\rightarrow Elim). The proof will be of the following format:

$$\frac{\Gamma \vdash A_1 \rightarrow \dots \rightarrow A_n \rightarrow A \rightarrow B \quad \forall i \leq n. \Gamma \vdash A_i}{\Gamma \vdash A \rightarrow B} \rightarrow\text{ELIM}$$

We can always provide an alternative proof that uses (\rightarrow Intro) first and looks like this:

$$\frac{\frac{\Gamma \vdash A_1 \rightarrow \dots \rightarrow A_n \rightarrow A \rightarrow B \quad \forall i \leq n. \Gamma, A \vdash A_i \quad \Gamma, A \vdash A}{\Gamma, A \vdash B} \rightarrow\text{ELIM}}{\Gamma \vdash A \rightarrow B} \rightarrow\text{INTRO}$$

The case where $\Gamma \vdash A \rightarrow B$ is proved by assumption is subsumed by the next answer.

- Proof by assumption is just a special case of (\rightarrow Elim) where $n = 0$. However, if we drop the assumption rule, proofs can now have a slightly different structure because we try (\rightarrow Intro) first:

$$\frac{\frac{A_1 \rightarrow \dots \rightarrow A_n \rightarrow B \in \Gamma' \quad \forall i \leq n. \Gamma' \vdash A_i}{\Gamma' \vdash B} \rightarrow\text{ELIM}}{\Gamma, A_1 \rightarrow \dots \rightarrow A_n \rightarrow B \vdash A_1 \rightarrow \dots \rightarrow A_n \rightarrow B} \rightarrow\text{INTRO } n \text{ TIMES}$$

with

$$\Gamma' := \Gamma, A_1 \rightarrow \dots \rightarrow A_n \rightarrow B, A_1, \dots, A_n.$$

- See [prover_12_sol.hs](#).

Exercise 2 (Intuitionistic Proofs)

Prove the following propositions in intuitionistic logic:

- a) $(A \rightarrow A) \vee B$
- b) $A \rightarrow (B \rightarrow A \wedge B)$
- c) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$

Solution

- a) Term: $\text{inl } (\lambda x. x)$. Proof:

$$\frac{\frac{\overline{A \vdash A}}{\vdash A \rightarrow A} \rightarrow\text{I}}{\vdash (A \rightarrow A) \vee B} \vee\text{I}_1$$

- b) Term: $\lambda x y. \langle x, y \rangle$. Proof:

$$\frac{\frac{\frac{\overline{A, B \vdash A} \quad \overline{A, B \vdash B}}{A, B \vdash A \wedge B} \wedge\text{I}}{A \vdash B \rightarrow (A \wedge B)} \rightarrow\text{I}}{\vdash A \rightarrow B \rightarrow (A \wedge B)} \rightarrow\text{I}$$

- c) Term: $\lambda x y z. \text{case } z \text{ of inl } a \Rightarrow x a \mid \text{inr } b \Rightarrow y b$. Proof:

$$\frac{\frac{\overline{\Gamma \vdash A \vee B} \quad \frac{\overline{\Gamma, A \vdash A \rightarrow C} \quad \overline{\Gamma, A \vdash A}}{\Gamma, A \vdash C} \rightarrow\text{E} \quad \frac{\overline{\Gamma, B \vdash B \rightarrow C} \quad \overline{\Gamma, B \vdash B}}{\Gamma, B \vdash C} \rightarrow\text{E}}{\Gamma := (A \rightarrow C), (B \rightarrow C), A \vee B \vdash C} \vee\text{E}}{\frac{\frac{\frac{\Gamma := (A \rightarrow C), (B \rightarrow C), A \vee B \vdash C}{(A \rightarrow C), (B \rightarrow C) \vdash (A \vee B) \rightarrow C} \rightarrow\text{I}}{(A \rightarrow C) \vdash (B \rightarrow C) \rightarrow (A \vee B) \rightarrow C} \rightarrow\text{I}}{\vdash (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow (A \vee B) \rightarrow C} \rightarrow\text{I}}$$

Homework 3 (Weak Normalization with Pairs)

We previously proved (Exercise 8.2) that every type-correct λ^{\rightarrow} -term has a β -normal form. Adapt the proof to accommodate for the extension of the simply typed lambda calculus with pairs.

Homework 4 (From Proof Terms to Propositions)

Consider the following proof term:

$$\lambda q. \lambda p. \text{case } \pi_1 p \text{ of } \text{inl } a \Rightarrow \text{inl } \langle \pi_1 q, \langle a, \pi_2 p \rangle \rangle \mid \text{inr } b \Rightarrow \text{inr } \langle \pi_2 q, b \rangle$$

- a) Exhibit the proposition that is proved by this term.
- b) Give the corresponding proof tree.

Homework 5 (Intuitionistic Proofs)

Prove the following propositions in pure logic, without lambda-terms, and write down the λ -term corresponding to each proof:

- a) $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$
- b) $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$