Logic Exercises

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Here is a website for syntax trees and truth tables.

Exercise 1.1. **[Hello Logic]**
Discuss: What does logic mean to you? Is it worth studying? Why? Why not? Where do we use logic? How did it come into being? What makes logic special?

Exercise 1.2. **[Basics]**
Let M be a set of formulas, and let F and G be formulas. Which of the following assertions hold?

1. If F satisfiable then $M \models F$
2. F is valid iff $\top \models F$
3. If $\models F$ then $M \models F$
4. If $M \models F$ then $M \cup \{G\} \models F$
5. $M \models F$ and $M \models \neg F$ cannot hold simultaneously
6. If $M \models G \to F$ and $M \models G$ then $M \models F$

Solution:
Assertions 2, 3, 4, and 6 hold.

For 4 note that $M \models F$ iff $\forall \mathcal{A}.\ (\forall H \in M.\ \mathcal{A} \models H) \implies \mathcal{A} \models F$.

Counterexample for 1: $F = A_1, M = \{A_2\}$

Counterexample for 5: $M = \{\bot\}$ *(ex falso quodlibet)*

Exercise 1.3. [Coincidence Lemma]
Assume that for all atomic formulas A_i in F, $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$. Show that

$$\mathcal{A} \models F \text{ iff } \mathcal{A}' \models F$$

Solution:
Proof by induction over the structure of F. Let $atoms(F)$ denote the set of all atomic formulas A_i in a formula F.

- Case $F = A_i$ for some i: $\mathcal{A} \models A_i \iff \mathcal{A}(A_i) = 1 = \mathcal{A}'(A_i) \iff \mathcal{A}' \models A_i$ (equality of assignments by assumption)
- Case $F = \neg G$ for some G:
  IH: $\mathcal{A} \models G \iff \mathcal{A}' \models G$
  Proof: $\mathcal{A} \models \neg G \iff \mathcal{A} \not\models G \overset{IH}{\iff} \mathcal{A}' \not\models G \iff \mathcal{A}' \models \neg G$
- Case $F = G \wedge H$ for some G, H:
  Observation: $atoms(F) = atoms(G) \cup atoms(H)$
  Hence, $\mathcal{A}$ and $\mathcal{A}'$ coincide on G and H too.
  We can thus obtain:
  IH 1: $\mathcal{A} \models G$ iff $\mathcal{A}' \models G$
  IH 2: $\mathcal{A} \models H$ iff $\mathcal{A}' \models H$
  Remaining proof trivial.

Exercise 1.4. [Anti-Interpolant]
Assume F and G do not share any atoms. Show that if $\models F \to G$ then F is unsatisfiable or G is a tautology (or both). *Hint:* you may want to use the previous result.

Solution:
Proof by contraposition. Assume that F is satisifiable *and* G is not a tautology. Obtain assignments $\mathcal{A}_F$ and $\mathcal{A}_G$ such that $\mathcal{A}_F \models F$ and $\mathcal{A}_G \not\models G$. Construct a new assignment $\mathcal{A}$ as follows:

$$\mathcal{A}(A_i) = \begin{cases} \mathcal{A}_F(A_i) & \text{if } A_i \in atoms(F) \\ \mathcal{A}_G(A_i) & \text{if } A_i \in atoms(G) \\ 0 & \text{otherwise} \end{cases}$$

This is well-defined, because $atoms(F) \cap atoms(G) = \emptyset$. $\mathcal{A}$ coincides with $\mathcal{A}_F$ on F and with $\mathcal{A}_G$ on G. By the coincidence lemma, $\mathcal{A} \models F$ and $\mathcal{A} \not\models G$. Hence $\mathcal{A} \not\models F \to G$ and thus $\not\models F \to G$.

Exercise 1.5. [Sense and Reference]
Pick an assignment $\mathcal{W}$. Call this assignment *the world*. Now pick a formula F. Then either $\mathcal{W} \models F \leftrightarrow \top$ or $\mathcal{W} \models F \leftrightarrow \bot$. Hence, each formula F under $\mathcal{W}$ is equal to $\top$ or $\bot$.

Discuss: Do you agree? For example, should we treat $F \vee \neg F$ as being equal to $\top$? Do both hold the same cognitive value?

Homework: Homework exercises will not be graded. Rather, you can ask for help and discuss the exercises and your solutions on Zulip.

Homework 1.1. [CNF and DNF] (+)

Use the rewriting-based procedure from the lecture to convert the following formulas F and G first to NNF, and then to CNF and DNF. Document each rewriting step.

$$F = \neg\neg(\neg A_1 \wedge \neg\neg(A_2 \vee A_3)) \qquad G = (A_1 \vee A_2 \vee A_3) \wedge (\neg A_1 \vee \neg A_2)$$

Solution:

Algorithmic.

Homework 1.2. [Basic equivalences] (+)

Let F and G be formulas. Are the following statements equivalent? Proof or counterexample!

1. $\models F \leftrightarrow G$
2. $F \equiv G$

What is the difference between $F \leftrightarrow G$ and $F \equiv G$?
How about these two statements? Prove or disprove!

1. F is valid
2. $F \equiv \top$

Solution:

They are equivalent: Assume $\models F \leftrightarrow G$ and let $\mathcal{A}$ be arbitrary. By assumption, either $\mathcal{A}(F \wedge G)$ or $\mathcal{A}(\neg F \wedge \neg G)$. In any case, $\mathcal{A}(F) = \mathcal{A}(G)$ and hence $F \equiv G$; other direction similar.
$F \leftrightarrow G$ is a formula *of* propositional logic while $F \equiv G$ is a mathematical statement *about* two propositional formulas.
The final two statements are also equivalent.

Homework 1.3. [Efficient CNF satisfiability check] (++)

In general, solving satisfiability for CNF formula is a hard problem. Consider the special case where clauses may only contain up to two literals. Give a polynomial time algorithm to check for satisfiability.

Solution:

See here.

Homework 1.4. **[Craig-Interpolant]** (+++)
Let F and G be arbitrary formulas with $F \models G$. Show that there is a formula H mentioning only propositional variables occuring in both F and G such that $F \models H$ and $H \models G$.

Solution:
Let $Var(F)$ and $Var(G)$ be the sets of propositional variables appearing in F and G, respectively. A truth table over the set of variables $Var(F) \cap Var(G)$ has a line for each assignment with domain $Var(F) \cap Var(G)$. Consider such a table for which the line corresponding to an assignment $\mathcal{A}$ has entry 1 iff $\mathcal{A}$ extends to a model $\mathcal{A}'$ on $Var(F)$ of F. Let H be a formula over variables $Var(F) \cap Var(G)$ that realises the above truthtable (e.g. take the CNF of the table).

Clearly, $F \models H$ (hint: take an assignment of F and consider its restriction to $Var(F) \cap Var(G)$).

To show that $H \models G$, suppose that $\mathcal{A}$ is a model of H. Then, by construction of H, there is an assignment $\mathcal{A}'$ which differs from $\mathcal{A}$ only on $Var(F) \setminus Var(G)$ such that $\mathcal{A}'(F) = 1$. Since $F \models G$, we have $\mathcal{A}'(G) = 1$. Since $\mathcal{A}$ and $\mathcal{A}'$ agree on $Var(G)$, we have $\mathcal{A}(G) = 1$.

> There can be no doubt that the knowledge of logic is of considerable practical importance for everyone who desires to think and infer correctly.
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> — Alfred Tarski