

LOGIC EXERCISES

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EXERCISE SHEET 5

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Exercise 5.1. [Induction Proofs]

Show that the following schema has a proof in natural deduction for all $n \geq 1$:

$$P_n = ((A_1 \wedge (A_2 \wedge (\dots \wedge A_n) \dots)) \rightarrow B) \rightarrow (A_1 \rightarrow (A_2 \rightarrow (\dots (A_n \rightarrow B) \dots))))$$

Exercise 5.2. [From Sequent Calculus to Natural Deduction]

How can we construct a natural deduction proof $\Gamma \vdash_N \bigvee \Delta$ from a sequent calculus proof $\Gamma \Rightarrow \Delta$?

Exercise 5.3. [Hilbert Calculus]

Prove the following formula with a linear proof in Hilbert calculus: $(F \wedge G) \rightarrow (G \wedge F)$

Hint: Use the deduction theorem.

Homework 5.1. [Small Hilbert] (++)

In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

$$\mathbf{A1} \quad F \rightarrow (G \rightarrow F)$$

$$\mathbf{A2} \quad (F \rightarrow G \rightarrow H) \rightarrow (F \rightarrow G) \rightarrow F \rightarrow H$$

$$\mathbf{A10} \quad (\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$$

Derive the following statement from the axioms above with the help of \rightarrow_E :

$$\neg(F \rightarrow F) \rightarrow G$$

Bonus: In fact, **Meredith** showed that all that is needed is one single axiom:

$$((((A \rightarrow B) \rightarrow (\neg C \rightarrow \neg D)) \rightarrow C) \rightarrow E) \rightarrow ((E \rightarrow A) \rightarrow (D \rightarrow A))$$

Try to derive some an axiom of your choice presented in the lecture in Meredith's system.

Homework 5.2. [From Sequent Calculus to Natural Deduction: Reloaded] (++)

In Exercise 5.2, we constructed a natural deduction proof $\Gamma \vdash_N \bigvee \Delta$ from a sequent calculus proof of $\Gamma \Rightarrow \Delta$. That construction created classical proofs because it required the use of the (\perp) rule.

Let us consider yet another restricted Sequent Calculus called "G3c". In G3c, we have $\Delta = \{F\}$, that is the succedent always contains exactly one formula. Here are the axioms:

$$\begin{array}{ll} \mathbf{Ax} \quad P, \Gamma \Rightarrow P \text{ (} P \text{ atomic)} & \mathbf{L\perp} \quad \perp, \Gamma \Rightarrow A \\ \mathbf{L\wedge} \quad \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} & \mathbf{R\wedge} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \\ \mathbf{L\vee} \quad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} & \mathbf{R\vee} \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \text{ (} i = 0, 1 \text{)} \\ \mathbf{L\rightarrow} \quad \frac{A \rightarrow B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \rightarrow B, \Gamma \Rightarrow C} & \mathbf{R\rightarrow} \quad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \end{array}$$

Give a direct construction that transforms a G3c proof $\Gamma \Rightarrow F$ into a natural deduction proof $\Gamma \vdash_N F$ without using the (\perp) rule. You are, however, allowed to use the intuitionistic rule $\perp \vdash_N F$; call it $(\perp E)$.

Homework 5.3. [Simulating Truth Tables](+)

In the lecture, the following lemma was discussed:

Let $\text{atoms}(F) \subseteq \{A_1, \dots, A_n\}$. Then we can construct a proof $A_1^A, \dots, A_n^A \vdash_N F^A$.

Recall the definition of F^A :

$$F^A = \begin{cases} F, & \text{if } \mathcal{A}(F) = 1 \\ \neg F, & \text{otherwise} \end{cases}$$

The proof proceeded by induction on F . The cases for atomic formulas as well as implication were shown in the lecture. Prove the cases for negation and disjunction!

Simplicity is the ultimate sophistication.

— William Gaddis