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Chair for Logic and Verification

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Exercise 6.1. [Equivalence]
Let F and G be arbitrary formulas (in particular, they may contain free occurrences of x). Which of the following equivalences hold? Proof or counterexample!

1. $\forall x(F \wedge G) \equiv \forall xF \wedge \forall xG$
2. $\exists x(F \wedge G) \equiv \exists xF \wedge \exists xG$

Solution:

1. Holds. Assume $\mathcal{A} \models \forall x(F \wedge G)$,
   $\iff$ for all $d \in U_{\mathcal{A}}$, we have $\mathcal{A}[d/x] \models F$ and $\mathcal{A}[d/x] \models G$,
   $\iff$ for all $d_1 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_1/x] \models F$ and for all $d_2 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_2/x] \models G$
   $\iff$ $\mathcal{A} \models \forall xF \wedge \forall xG$
2. Does not hold. Let $F = P(x)$ and $G = Q(x)$, $U_{\mathcal{A}} = \{0, 1\}$, $P^{\mathcal{A}} = \{0\}$, and $Q^{\mathcal{A}} = \{1\}$. Clearly, $\mathcal{A} \models \exists xF \wedge \exists xG$ but $\mathcal{A} \not\models \exists x(F \wedge G)$

Exercise 6.2. [Skolem Form]
Convert the following formula into a rectified formula, closed and rectified formula, RPF and Skolem form (in order as given).

$$P(x) \wedge \forall x \ (Q(x) \wedge \forall x \exists y \ P(f(x, y)))$$

Solution:

$$\begin{aligned}
&P(x) \wedge \forall x(Q(x) \wedge \forall x \exists y P(f(x, y))) && \\
\rightsquigarrow &P(x) \wedge \forall x_1(Q(x_1) \wedge \forall x_2 \exists y P(f(x_2, y))) && \text{rectified} \\
\rightsquigarrow &\exists x P(x) \wedge \forall x_1(Q(x_1) \wedge \forall x_2 \exists y P(f(x_2, y))) && \text{rectified and closed} \\
\rightsquigarrow &\exists x \forall x_1 \forall x_2 \exists y(P(x) \wedge (Q(x_1) \wedge P(f(x_2, y)))) && \text{RPF} \\
\rightsquigarrow &\forall x_1 \forall x_2(P(g) \wedge (Q(x_1) \wedge P(f(x_2, h(x_1, x_2))))) && \text{Skolem form}
\end{aligned}$$

Exercise 6.3. [**Tarski's World**]
Axiomatise the following statements using all predicates allowed in Tarski's world on the lecture website except equality. Moreover, do not use any free variables nor constant symbols.

1. No square is between two other elements.
2. From left to right elements are ordered descendingly according to their size.
3. No pentagon is smaller than any other element.
4. Every square is between two triangles.
5. There is nothing between two squares.

Solution:

1. $\forall x \forall y \forall z(\text{Between}(x, y, z) \rightarrow \neg\,\text{Square}(x))$
2. $\forall x \forall y(\text{LeftOf}(x, y) \rightarrow \text{Greater}(x, y))$
3. $\forall x \forall y((\text{Pentagon}(x) \rightarrow \neg\,\text{Smaller}(x, y))$
4. $\forall x(\text{Square}(x) \rightarrow \exists y \exists z(\text{Triangle}(y) \wedge \text{Triangle}(z) \wedge \text{Between}(x, y, z)))$
5. $\forall x \forall y((\text{Square}(x) \wedge \text{Square}(y)) \rightarrow \neg\exists z\,\text{Between}(z, x, y))$

Homework 6.1. **[Orders]** (+)

A reflexive and transitive relation is called a *preorder*. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z\ (P(x,x) \wedge (P(x,y) \wedge P(y,z) \rightarrow P(x,z))).$$

1. Which of the following structures are models of F? Give an informal proof in the positive case and a counterexample for the negative case!
   (a) $U^{\mathcal{A}} = \mathbb{N}$ and $P^{\mathcal{A}} = \{(m,n) \mid m > n\}$
   (b) $U^{\mathcal{A}} = \mathbb{Z} \times \mathbb{Z}$ and $P^{\mathcal{A}} = \{((x,y),(a,b)) \mid\ a - x \leq b - y\ \}$
   (c) $U^{\mathcal{A}} = \mathbb{R}$ and $P^{\mathcal{A}} = \{(m,n) \mid m = n\}$
2. Let $Q(x,y)$ be specified as follows: $\forall x \forall y (P(x,y) \leftrightarrow Q(y,x))$. Assuming P is a preorder, is Q also a preorder?
3. Specify the notion of *equivalence relations*, that is, preorders that additionally satisfy symmetry.

Solution:

1. (a) No, it is not reflexive
   (b) Yes. The relation is obviously reflexive. Now assume $(c_1,d_1),(c_2,d_2),(c_3,d_3)$, $c_2 - c_1 \leq d_2 - d_1$ and $c_3 - c_2 \leq d_3 - d_2$. Then
   $$c_3 - c_1 \leq d_3 - d_2 + c_2 - c_1 \leq d_3 - d_2 + d_2 - d_1 = d_3 - d_1$$
   Hence it also is transitive.
   (c) Trivially yes.
2. Yes. We have $\forall x.\, P(x,x)$ and hence $\forall x.\, Q(x,x)$. Assume $Q(a,b)$ and $Q(b,c)$. Then we have $P(b,a)$ and $P(c,b)$, thus $P(c,a)$, and hence $Q(a,c)$.
3. Add the conjunct $P(x,y) \rightarrow P(y,x)$ to F.

Homework 6.2. **[Skolem Form]** (+)

Convert the following formulas into a rectified formula, closed and rectified formula, RPF and Skolem form (in order as given).

1. $\forall x \exists y \forall z \exists w (\neg Q(f(x),y) \wedge P(a,w))$
2. $\forall z (\exists y (P(x,g(y),z)) \vee \neg \forall x\ Q(x))$

Homework 6.3. **[Tarski's World Reloaded]** (++)
Create structures that satisfy each of the following set of formulas in Tarski's world. You can download configuration files containing these formulas and load them into the applet here and here.

1.

$$\forall x \forall y(\text{Triangle}(x) \land \text{Triangle}(y) \rightarrow x = y)$$
$$\exists x(\text{Triangle}(x) \land \text{Medium}(x))$$
$$\text{Square}(a) \land \text{Square}(b) \land \neg\exists x(\text{Between}(x, a, b))$$
$$\forall x \forall y((\text{Pentagon}(x) \land \text{Pentagon}(y) \land \neg(x = y)) \rightarrow \neg \text{SameSize}(x, y))$$
$$\text{Pentagon}(c) \land \exists y(\text{SameRow}(c, y) \land \text{Square}(y))$$
$$\forall x(\text{Between}(x, d, e) \rightarrow \text{Pentagon}(x))$$
$$\exists x \exists y \exists z(\text{Between}(x, d, e) \land \neg(x = y) \land \neg(y = z) \land \neg(x = z))$$
$$\text{SameRow}(b, d)$$
$$\text{SameCol}(a, d)$$
$$\forall x(\text{SameCol}(x, d) \land \neg(x = d) \rightarrow x = a)$$

2.

$$\forall x \exists y \neg \text{SameSize}(x, y)$$
$$\forall x \forall y(\text{Triangle}(x) \land \text{Triangle}(y) \land \text{SameCol}(x, y) \rightarrow (x = y))$$
$$\forall x \forall y(\text{Smaller}(x, y) \land \text{SameRow}(x, y) \rightarrow \text{LeftOf}(x, y))$$
$$\text{Pentagon}(b) \land \text{Medium}(b)$$
$$\forall x(\text{Square}(x) \leftrightarrow \text{SameSize}(x, c))$$
$$\exists x \exists y(\text{Triangle}(x) \land \text{Square}(y) \land \text{Between}(x, a, y))$$
$$\exists x \exists y(\text{Triangle}(x) \land \text{Square}(y) \land \text{SameCol}(x, y))$$
$$\text{Between}(b, a, c)$$
$$\forall x(\text{Pentagon}(x) \rightarrow \exists y(\text{Pentagon}(y) \land \neg(x = y) \land \text{SameCol}(x, y)))$$
$$\forall x(\text{Square}(x) \land \text{Small}(x) \rightarrow (x = a))$$

Solution:
Here and here

Homework 6.4. [Kindermund] (+)

The kindergarten teachers at the Garching campus came up with a strategy to improve the children's discipline. They promised a prize for those who behave well. Here is what they announced:

1. All kids who do their homework will receive a cake.
2. Every kid that does not start a fight against any other kid will receive a cake.
3. There is (at least) one kid who does the homework and against whom no other kid starts a fight.

One kid concluded the following: every kid that received a cake, did not start a fight with any other kid.

Prove that this conclusion is wrong. Give three formulas in first-order logic, T_1, T_2, T_3 that represent the statements given by the teachers and one formula K that represents the conclusion by the kid. Use predicate symbols H, F, C for doing homework, starting a fight with someone, and receiving a cake, respectively. Show that $T_1, T_2, T_3 \not\models K$ by specifying a finite countermodel.

Solution:

The statements can be modelled as follows:

$$\begin{aligned} T_1 &= \forall x(H(x) \to C(x)) \\ T_2 &= \forall x((\forall y \neg F(x, y)) \to C(x)) \\ T_3 &= \exists x(H(x) \wedge \forall y \neg F(y, x)) \\ K &= \forall x(C(x) \to \neg \exists y F(x, y)) \end{aligned}$$

We can take a structure with two kids, $U_{\mathcal{A}} = \{Alice, Bob\}$, with $H^{\mathcal{A}} = \{\text{Bob}\}$, $C^{\mathcal{A}} = \{\text{Alice}, \text{Bob}\}$, and $F^{\mathcal{A}} = \{(\text{Bob}, \text{Alice})\}$. In this case we have $\mathcal{A}(T_1) = \mathcal{A}(T_2) = \mathcal{A}(T_3) = 1$ and $\mathcal{A}(K) = 0$.

To be is to be the value of a variable.

— Willard Van Orman Quine