Logic Exercises

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Exercise 7.1. [(In)finite Models]
Consider predicate logic with equality. We use infix notation for equality and abbreviate $\neg(s = t)$ by $s \neq t$. Moreover, we call a structure finite if its universe is finite.

1. Specify a finite model for the formula $\forall x\, (c \neq f(x) \wedge x \neq f(x))$.
2. Specify a model for the formula $\forall x \forall y\ (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$.
3. Show that the second formula has no finite model.

Exercise 7.2. [Herbrand Structures]
Consider the formula

$$F = \forall x \forall y (P(f(x), g(y)) \wedge \neg P(g(x), f(y)))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F that is not a model of F.

Exercise 7.3. [Ground Resolution]
Use ground (Gilmore) resolution to prove that the following formula is valid:

$$(\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)$$

Exercise 7.4. [Uncountable "Natural Numbers"]
We consider the following axioms in an attempt to model the natural numbers in first-order logic with equality:

1. $F_1 = \forall x \forall y (f(x) = f(y) \rightarrow x = y)$
2. $F_2 = \forall x (f(x) \neq 0)$
3. $F_3 = \forall x (x = 0 \vee \exists y (x = f(y)))$

Give a model with an *uncountable* universe for:

1. $\{F_1, F_2\}$
2. $\{F_1, F_2, F_3\}$

Remember: A set S is uncountable if there is no bijection between S and $\mathbb{N}$.

Homework 7.1. [Model Sizes] (++)

1. Specify a satisfiable formula F (one with and one without equality) such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \geq 4$.
2. Can you also specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \leq 4$? Again, consider both predicate logic with and without equality.
3. Specify a satisfiable formula F with equality such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \in 2\mathbb{N}_{>0}$.

Homework 7.2. [Herbrand Structures] (+)

Consider the formula

$$F = \forall x(P(f(x)) \leftrightarrow \neg P(x))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F that is not a model of F.

Homework 7.3. [Preconditions Are Here To Stay] (+)

Recall the fundamental theorem from the lecture: "Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model".

Explain: what goes wrong if the precondition is violated, that is when F is not closed or not in Skolem form. Describe both cases.

Homework 7.4. [Ground resolution] (++)

Execute ground resolution to show that the following formula is unsatisfiable:

$$\forall x \forall y((P(x) \wedge \neg Q(y,y)) \rightarrow Q(x,y)) \wedge \neg \exists x(P(x) \wedge \exists y(Q(y,y) \wedge Q(x,y))) \wedge \exists y(P(y))$$

Homework 7.5. [Proof of the Fundamental Theorem] (++)

Recall the fundamental theorem: Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model. Give the omitted proof for the base case (slide 6, $\mathcal{A}(G) = \mathcal{T}(G)$).

> Logic takes care of itself; all we have to do is to look and see how it does it.
> — Ludwig Wittgenstein