Prof. Tobias Nipkow
Kevin Kappelmann

Exercise 7.1. [(In)finite Models]
Consider predicate logic with equality. We use infix notation for equality and abbreviate $\neg(s = t)$ by $s \neq t$. Moreover, we call a structure finite if its universe is finite.

1. Specify a finite model for the formula $\forall x\,(c \neq f(x) \wedge x \neq f(x))$.
2. Specify a model for the formula $\forall x \forall y\ (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$.
3. Show that the second formula has no finite model.

Solution:

1. $U^{\mathcal{A}} = \{0, 1, 2\} \subset \mathbb{N}$, $c^{\mathcal{A}} = 0$, $f^{\mathcal{A}}(0) = 1$, and $f^{\mathcal{A}}(n+1) = 2 - n$
2. $U^{\mathcal{A}} = \mathbb{N}$ and $c^{\mathcal{A}} = 0$ and $f^{\mathcal{A}}(n) = n + 1$
3. Assume $\mathcal{M}$ is a finite model of the formula. By the second conjunct, $f^{\mathcal{M}}$ is injective so $|U_{\mathcal{M}}| \leq |f(U_{\mathcal{M}})|$. Further, $f(U_{\mathcal{M}}) \subseteq U_{\mathcal{M}}$ so $|f(U_{\mathcal{M}})| = |U_{\mathcal{M}}|$ and hence $f(U_{\mathcal{M}}) = U_{\mathcal{M}}$ (using our finiteness assumption). Thus there is $d \in U_{\mathcal{M}}$ such that $f(d) = c^{\mathcal{M}}$, contradicting the first conjunct.

Exercise 7.2. [Herbrand Structures]
Consider the formula

$$F = \forall x \forall y (P(f(x), g(y)) \wedge \neg P(g(x), f(y)))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F that is not a model of F.

Solution:
We define $U_{\mathcal{A}} = T(F)$, i.e., the Herbrand universe for F. We invent a constant $a \in T(F)$. We define $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ to be the Herbrand interpretations.

1. $P^{\mathcal{A}} = \{(f(t_1), g(t_2)) \mid t_1, t_2 \in T(F)\}$.
2. $P^{\mathcal{A}} = \{(g(t_1), f(t_2)) \mid t_1, t_2 \in T(F)\}$.

Exercise 7.3. [Ground Resolution]
Use ground (Gilmore) resolution to prove that the following formula is valid:

$$(\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)$$

Solution:

First put the formula into Skolem form:

$$\begin{aligned}
&\neg((\forall x P(x, f(x))) \longrightarrow \exists y P(c, y)) && \\
&(\forall x P(x, f(x))) \land \neg\exists y P(c, y)) && \text{(push negation)} \\
&(\forall x P(x, f(x))) \land \forall y \neg P(c, y)) && \text{(push negation)} \\
&\forall x \forall y (P(x, f(x)) \land \neg P(c, y)) && \text{(Skolem-Form)}
\end{aligned}$$

Now enumerate the Herbrand expansion:

$$CE(F) = \{P(c, f(c)), \neg P(c, f(c)), \ldots\}$$

With resolution, we immediately get $\Box$ from the first two items in the enumeration.

Exercise 7.4. **[Uncountable "Natural Numbers"]**
We consider the following axioms in an attempt to model the natural numbers in first-order logic with equality:

1. $F_1 = \forall x \forall y (f(x) = f(y) \to x = y)$
2. $F_2 = \forall x (f(x) \neq 0)$
3. $F_3 = \forall x (x = 0 \lor \exists y (x = f(y)))$

Give a model with an *uncountable* universe for:

1. $\{F_1, F_2\}$
2. $\{F_1, F_2, F_3\}$

Remember: A set S is uncountable if there is no injection from S to $\mathbb{N}$.

Solution:

1. $U_{\mathcal{A}} = \mathbb{R}_0^+$, $0^{\mathcal{A}} = 0$, and $f^{\mathcal{A}}(x) = x + 1$
   $f^{\mathcal{A}}$ is clearly injective and there is no x such that $f^{\mathcal{A}}(x) = 0$, because $-1 \notin U_{\mathcal{A}}$.
2. We take $U_{\mathcal{A}}$ to be the union of the positive real numbers and the non-positive whole numbers, i.e., $U_{\mathcal{A}} = \mathbb{R}_{>0} \cup \mathbb{Z}_{\leq 0}$.
   Let the symbols be interpreted as follows:

$$0^{\mathcal{A}} = 0$$
$$f^{\mathcal{A}}(x) = \begin{cases} 2x & \text{if } x > 0 \\ x - 1 & \text{if } x \leq 0 \end{cases}$$

   (a) $f^{\mathcal{A}}$ is defined as two disjoint domains that have disjoint ranges and f is injective on both domains; hence the entire function is injective.
   (b) 0 is not in the range of $f^{\mathcal{A}}$: For $x > 0$, $f^{\mathcal{A}}(x) > 0$ and for $x \leq 0$, $f^{\mathcal{A}}(x) \leq -1$.
   (c) To show: $x \neq 0 \to \exists y (x = f(y))$.
   If $x < 0$, then $x \leq -1$, hence $x = f^{\mathcal{A}}(x + 1)$.
   Otherwise, $x = f^{\mathcal{A}}\left(\frac{x}{2}\right)$.

Homework 7.1. **[Model Sizes]** (++)

1. Specify a satisfiable formula F (one with and one without equality) such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \geq 4$.
2. Can you also specify a satisfiable formula F such that for all models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \leq 4$? Again, consider both predicate logic with and without equality.
3. Specify a satisfiable formula F with equality such that for all finite models $\mathcal{A}$ of F, we have $|U_{\mathcal{A}}| \in 2\mathbb{N}_{>0}$.

Solution:

1. $\exists x_1, x_2, x_3, x_4.\, x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge x_3 \neq x_4$
   Without equality, one can use two predicates B_1 and B_0 representing bits and 4 constants a_0, a_1, a_2, a_3 and for each a_i, encode i in binary using B_1 and B_0, i.e. $\neg B_1(a_0) \wedge \neg B_1(a_0) \wedge \neg B_1(a_1) \wedge B_1(a_1) \wedge B_1(a_2) \wedge \neg B_1(a_2) \cdots$.
2. $\forall x \forall y.\, x = y$

   Without equality, no limit on the model size can be given: assume there is a finite model $\mathcal{M}$. Take an arbitrary element e of $\mathcal{M}$. We can add a new element e_1 that behaves exactly the same as e to $\mathcal{M}$ without changing the set of formula satisfied by $\mathcal{M}$ (proof by structural induction on formulas). We continue this process iteratively to obtain a model of arbitrary, countable size. Indeed, we could even add an infinite number of copies of e and obtain models of arbitrary size.
3. $\forall x\, \neg P(x,x) \wedge \forall x \exists y P(x,y) \wedge \forall x \forall y \forall z (P(x,y) \wedge P(y,z) \rightarrow (x = z))$

Homework 7.2. **[Herbrand Structures]** (+)

Consider the formula

$$F = \forall x (P(f(x)) \leftrightarrow \neg P(x))$$

1. Specify a Herbrand model for F.
2. Specify a Herbrand structure suitable for F that is not a model of F.

Homework 7.3. **[Preconditions Are Here To Stay]** (+)

Recall the fundamental theorem from the lecture: "Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model".

Explain: what goes wrong if the precondition is violated, that is when F is not closed or not in Skolem form. Describe both cases.

Solution:

$\exists x.\, (P(x) \wedge \neg P(a))$; there is no Herbrand model because there is only the constant a and no functionals but we need at least two elements for the formula to be satisfiable. The same problems arises for $P(x) \wedge \neg P(a)$.

Homework 7.4. [Ground resolution] (++)

Execute ground resolution to show that the following formula is unsatisfiable:

$$\forall x \forall y((P(x) \land \neg Q(y,y)) \to Q(x,y)) \land \neg\exists x(P(x) \land \exists y(Q(y,y) \land Q(x,y))) \land \exists y(P(y))$$

Solution:

Algorithmic.

Homework 7.5. [Proof of the Fundamental Theorem] (++)

Recall the fundamental theorem: Let F be a closed formula in Skolem form. Then F is satisfiable iff it has a Herbrand model. Give the omitted proof for the base case (slide 6, $\mathcal{A}(G) = \mathcal{T}(G)$).

Solution:

Let $\mathcal{A}$ be an arbitrary model of F. We define a Herbrand structure $\mathcal{T}$ as follows (according to the lecture):

$$U_{\mathcal{T}} = T(F) \qquad f^{\mathcal{T}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$$
$$(t_1, \dots, t_n) \in P^{\mathcal{T}} \text{ iff } (\mathcal{A}(t_1), \dots, \mathcal{A}(t_n)) \in P^{\mathcal{A}}$$

Additionally, if F contains no constant: $a^{\mathcal{A}} = u$ for some arbitrary $u \in U_{\mathcal{A}}$.

We now prove the omitted case for the following stronger proposition: For every closed formula G in Skolem form such that all function and predicate symbols in G occur in F, if $\mathcal{A} \models G$ then $\mathcal{T} \models G$. The proof proceeds by induction on the number n of universal quantifiers in G.

- Base case: $n = 0$. G has no quantifiers (because it is in Skolem form).
  Claim: $\mathcal{A}(G) = \mathcal{T}(G)$.
  Proof by induction on the structure of G.
  - Base case: $G = P(t_1, \dots, t_k)$
    We know that $\mathcal{A}(\mathcal{T}(t)) = \mathcal{A}(t)$, because $\mathcal{T}(t) = t$.

$$\begin{aligned} \mathcal{T} \models P(t_1, \dots, t_k) &\text{ iff } (\mathcal{T}(t_1), \dots, \mathcal{T}(t_k)) \in P^{\mathcal{T}} \\ &\text{ iff } (\mathcal{A}(\mathcal{T}(t_1)), \dots, \mathcal{A}(\mathcal{T}(t_k))) \in P^{\mathcal{A}} \\ &\text{ iff } (\mathcal{A}(t_1), \dots, \mathcal{A}(t_k)) \in P^{\mathcal{A}} \\ &\text{ iff } \mathcal{A} \models P(t_1, \dots, t_k) \end{aligned}$$

  - Induction step: $G = H_1 \land H_2$
    Induction hypotheses: $\mathcal{A}(H_i) = \mathcal{T}(H_i)$ for $i \in \{1, 2\}$

$$\begin{aligned} \mathcal{A}(H_1 \land H_2) &\text{ iff } \mathcal{A}(H_1) \text{ or } \mathcal{A}(H_2) \\ &\text{ iff } \mathcal{T}(H_1) \text{ or } \mathcal{T}(H_2) \\ &\text{ iff } \mathcal{T}(H_1 \land H_2) \end{aligned}$$

  - Other induction steps are similar.

Logic takes care of itself; all we have to do is to look and see how it does it.
— Ludwig Wittgenstein