Hilbert Systems
Propositional Logic

(See the book by Troelstra and Schwichtenberg)

Easy to define, hard to use

No context management

A Hilber system for propositional logic consists of

- a set of axioms (formulae)
- and a single infrence rule, $\to E$ or modus ponens:

$$\frac{F \to G \quad F}{G} \to E$$

Proof trees for some Hilbert system are labeled with formulas. The only inference rule is $\to E$.

Definition

We write $\Gamma \vdash_H F$ if there is a proof tree with root F whose leaves are either axioms or elements of Γ.

Alternative proof presentation

Proofs in Hilbert systems are freqently shown as lists of lines

1. F_1 *justification*$_1$
2. F_2 *justification*$_2$

$\vdots$

i. F_i *justification*$_i$

$\vdots$

justification$_i$ is either
assumption, *axiom* or $\rightarrow E(j, k)$ where $j, k < i$

Like linearized tree but also allows sharing of subproofs

Notational convention:

$$F \rightarrow G \rightarrow H \quad \text{means} \quad F \rightarrow (G \rightarrow H)$$

Note: $F \rightarrow G \rightarrow H \quad \equiv \quad F \wedge G \rightarrow H$
$F \rightarrow G \rightarrow H \quad \not\equiv \quad (F \rightarrow G) \rightarrow H$

Example (A simple Hilbert system)

Axioms:

$$F \to (G \to F) \qquad (A1)$$

$$(F \to G \to H) \to (F \to G) \to F \to H \qquad (\mathrm{A2})$$

A proof of $F \to F$:

$$\dfrac{\dfrac{}{\to (F \to F)}\ {\to}E}{F \to F}\ {\to}E$$

$\Rightarrow\ \vdash_H F \to F$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \textit{iff} \quad \Gamma \vdash_H F \rightarrow G$$

Proof "$\Leftarrow$":

$\Gamma \vdash_H F \rightarrow G$

$\Rightarrow \ F, \Gamma \vdash_H F \rightarrow G$

$\Rightarrow \ F, \Gamma \vdash_H G$ by $\rightarrow E$ because $F, \Gamma \vdash_H F$

Theorem (Deduction Theorem)

In any Hilbert-system that contains the axioms A1 and A2:

$$F, \Gamma \vdash_H G \quad \mathit{iff} \quad \Gamma \vdash_H F \to G$$

Proof "$\Rightarrow$":
By induction on (the length/depth of) the proof of $F, \Gamma \vdash_H G$
Then by cases on the last proof step:

Case $G = F$: see proof of $F \to F$ from $A1$ and $A2$

Case $G \in \Gamma$ or axiom: by $A1$ and ...

Case $\to E$ from $H \to G$ and H:

$$\dfrac{\dfrac{(F \to H \to G) \to (F \to H) \to F \to G \qquad F \to H \to G}{(F \to H) \to F \to G} \qquad F \to H}{F \to G}$$

Hilbert System

From now on $\vdash_H$ refers to the following set of axioms:

$$
\begin{aligned}
&F \to G \to F && (A1)\\
&(F \to G \to H) \to (F \to G) \to F \to H && (A2)\\
&F \to G \to F \land G && (A3)\\
&F \land G \to F && (A4)\\
&F \land G \to G && (A5)\\
&F \to F \lor G && (A6)\\
&G \to F \lor G && (A7)\\
&F \lor G \to (F \to H) \to (G \to H) \to H && (A8)\\
&(\neg F \to \bot) \to F && (A9)
\end{aligned}
$$

Relating
Hilbert and Natural Deduction

Theorem (Hilbert can simulate ND)

If $\Gamma \vdash_N F$ *then* $\Gamma \vdash_H F$

Proof translation in two steps: $\vdash_N \quad \leadsto \quad \vdash_H \; + \rightarrow I \quad \leadsto \quad \vdash_H$

1. Transform a ND-proof tree into a proof tree containing
   Hilbert axioms, $\rightarrow E$ and $\rightarrow I$
   by replacing all other ND rules by Hilbert proofs incl. $\rightarrow I$
   Principle: ND rule $\leadsto$ 1 axiom $+ \rightarrow I/E$

2. Eliminate the $\rightarrow I$ rules by the Deduction Theorem

Lemma (ND can simulate Hilbert)

If $\Gamma \vdash_H F$ *then* $\Gamma \vdash_N F$

Proof by induction on $\Gamma \vdash_H F$.

- Every Hilbert axiom is provable in ND (Exercise!)
- $\rightarrow E$ is also available in ND

Corollary

$\Gamma \vdash_H F$ *iff* $\Gamma \vdash_N F$

Corollary (Soundness and completeness)

$\Gamma \vdash_H F$ *iff* $\Gamma \models F$