First-Order Logic
Undecidability

[Cutland, *Computability*, Section 6.5.]

- Aim:
  Show that validity of first-order formulas is undecidable
- Method:
  Reduce the halting problem to validity of formulas
  by expressing program behaviour as formulas

Logical formulas can talk about computations!

Register machine programs (RMPs)

A register machine program is a sequence of instructions $I_1, \ldots, I_t$.
The instructions manipulate registers R_i $(i = 1, 2, \ldots)$
that contain (unbounded!) natural numbers.
There are 4 instructions:

$$
\begin{array}{l}
R_n := 0 \\
R_n := R_n + 1 \\
R_n := R_m \\
\text{IF } R_m = R_n \text{ GOTO } p
\end{array}
$$

Assumption: all jumps in a program go to $1, \ldots, t+1$;
execution terminates when the PC is $t+1$.

Let r be the maximal index of any register used in a program P.
Then the state of P during execution can be described by a tuple of natural numbers

$$(n_1, \ldots, n_r, k)$$

where n_i is the contents of R_i and k is the PC (the number of the next instruction to be executed).

Undecidability

Theorem (Undecidability of the halting problem for RMPs)
It is undecidable if a given register machine program terminates when started in state $(0, \ldots, 0, 1)$.

We reduce the halting problem for RMPs to the validity problem for first-order formulas.

Notation:
$P(0)\downarrow$ = "RMP P started in state $(0, \ldots, 0, 1)$ terminates"

Theorem
Given an RMP P we can effectively construct a closed formula φ_P *such that* $P(0)\downarrow$ *iff* $\models \varphi_P$.

Proof by construction of φ_P from $P = I_1, \ldots, I_t$.
Funct. symb.: z, s. Abbr.: $\overline{0} = z$, $\overline{1} = s(z)$, $\overline{2} = s(s(z))$, ...
Pred. symb.: R (arity: $r+1$) "reachable"
Aim: if $R(\overline{n_1}, \ldots \overline{n_r}, \overline{k})$ then $(0, \ldots, 0, 1) \overset{P}{\rightsquigarrow} (n_1, \ldots, n_r, k)$
For every I_i construct closed formula Ψ_i:

$$I_i = (R_n := 0)\text{:}\ \Psi_i := \forall x_1 \ldots x_r\ \ (R(x_1, \ldots, x_n, \ldots, x_r, \bar{i}) \rightarrow R(x_1, \ldots, z, \ldots, x_r, s(\bar{i}))$$

$I_i = (R_n := r_n + 1)$: the same except $s(x_n)$ instead of z

$I_i = (R_n := 0)$: the same except x_m instead of z

$I_i = (IF\ \ R_m = R_n\ \ GOTO\, p)$:

$$\Psi_i := \forall x_1 \ldots x_r (R(x_1, \ldots, x_r, \bar{i}) \rightarrow (x_m = x_n \rightarrow R(x_1, \ldots, x_r, \overline{p})) \wedge (x_m \neq x_n \rightarrow R(x_1, \ldots, x_r, s(\bar{i})))$$

$$\Psi_P := \Psi \wedge R(z, \ldots, z, s(z)) \wedge \Psi_1 \wedge \cdots \wedge \Psi_t$$

Ψ enforces that every model is similar to $\mathbb{N}$:
$\Psi := \forall x \forall y (s(x) = s(y) \rightarrow x = y) \wedge \forall x (z \neq s(x))$
(How can models of Ψ differ from $\mathbb{N}$?)

$\varphi_P := \Psi_P \to \tau$ where $\tau := \exists x_1 \ldots x_r\, R(x_1, \ldots, x_n, s(\bar{t}))$

Claim: $P(0)\downarrow$ iff $\models \varphi_P$

"$\Rightarrow$": Assume $P(0)\downarrow$, show $\models \varphi_P$. Assume $\mathcal{A} \models \Psi_P$.

Lemma

If $(0, \ldots, 0, 1) \overset{P}{\leadsto} (n_1, \ldots, n_r, k)$ *then* $\mathcal{A} \models R(\overline{n_1}, \ldots, \overline{n_r}, \overline{k})$

Proof by induction on the length of the execution using $\mathcal{A} \models \Psi_P$.

Thus $\mathcal{A} \models \tau$ because $P(0)\downarrow$.

"$\Leftarrow$": $\models \varphi_P \;\Rightarrow\; \mathcal{N} \models \varphi_P \;\Rightarrow\; (\mathcal{N} \models \Psi_P \Rightarrow \mathcal{N} \models \tau) \;\Rightarrow\; P(0)\downarrow$
where $U_{\mathcal{N}} := \mathbb{N}$, $z^{\mathcal{N}} := 0$ $s^{\mathcal{N}}(n) := n + 1$,
$R^{\mathcal{N}} := \{s \mid (0, \ldots, 0, 1) \overset{P}{\leadsto} s\}$