Propositional Logic
Definitional CNF

Definitional CNF

The definitional CNF of a formula is obtained in 2 steps:

1. Repeatedly replace a subformula G of the form $\neg A'$, $A' \wedge B'$ or $A' \vee B'$ by a new atom A and conjoin $A \leftrightarrow G$.
   This replacement is not applied to the "definitions" $A \leftrightarrow G$ but only to the (remains of the) original formula.
2. Translate all the subformulas $A \leftrightarrow G$ into CNF.

Example

$$\boxed{\neg(A_1 \vee A_2) \wedge A_3}$$

$$\rightsquigarrow$$

$$\boxed{\neg A_4 \wedge A_3} \wedge (A_4 \leftrightarrow (A_1 \vee A_2))$$

$$\rightsquigarrow$$

$$\boxed{A_5 \wedge A_3} \wedge (A_4 \leftrightarrow (A_1 \vee A_2)) \wedge (A_5 \leftrightarrow \neg A_4)$$

$$\rightsquigarrow$$

$$A_5 \wedge A_3 \wedge CNF(A_4 \leftrightarrow (A_1 \vee A_2)) \wedge CNF(A_5 \leftrightarrow \neg A_4)$$

Definitional CNF: Complexity

Let the initial formula have size n.

1. Each replacement step increases the size of the formula by a constant.
   There are at most as many replacement steps as subformulas, linearly many.
2. The conversion of each $A \leftrightarrow G$ into CNF increases the size by a constant.
   There are only linearly many such subformulas.

Thus the definitional CNF has size $O(n)$.

Notation

Definition

The notation $F[G/A]$ denotes the result of replacing all occurrences of the atom A in F by G.
We pronounce it as "F with G for A".

Example

$(A \wedge B)[(A \rightarrow B)/B] = (A \wedge (A \rightarrow B))$

Definition

The notation $\mathcal{A}[v/A]$ denotes a modified version of $\mathcal{A}$ that maps A to v and behaves like $\mathcal{A}$ otherwise:

$$(\mathcal{A}[v/A])(A_i) = \begin{cases} v & \text{if } A_i = A \\ \mathcal{A}(A_i) & \text{otherwise} \end{cases}$$

Substitution Lemma

Lemma

$\mathcal{A}(F[G/A]) \;=\; \mathcal{A}'(F)$ *where $A' = A[A(G)/A]$*

Example

$\mathcal{A}((A_1 \wedge A_2)[G/A_2]) = \mathcal{A}'(A_1 \wedge A_2)$ where $\mathcal{A}' = \mathcal{A}[\mathcal{A}(G)/A_2]$

Proof by structural induction on F.
Case F is an atom:
If $F = A$: $\mathcal{A}(F[G/A]) = \mathcal{A}(G) = \mathcal{A}'(F)$
If $F \neq A$: $\mathcal{A}(F[G/A]) = \mathcal{A}(F) = \mathcal{A}'(F)$

Case $F = F_1 \wedge F_2$:
$\mathcal{A}(F[G/A]) =$
$\mathcal{A}(F_1[G/A] \wedge F_2[G/A]) =$
$min(\mathcal{A}(F_1[G/A]), \mathcal{A}(F_2[G/A])) \overset{IH}{=}$
$min(\mathcal{A}'(F_1), \mathcal{A}'(F_2)) = \mathcal{A}'(F_1 \wedge F_2) = \mathcal{A}'(F)$

Definitional CNF: Correctness

Each replacement step produces an equisatisfiable formula:

Lemma

Let A be an atom that does not occur in G.
Then $F[G/A]$ is equisatisfiable with $F \wedge (A \leftrightarrow G)$.

Proof If $F[G/A]$ is satisfiable by some assignment $\mathcal{A}$, then by the Substitution Lemma, $\mathcal{A}' = \mathcal{A}[\mathcal{A}(G)/A]$ is a model of F. Moreover $\mathcal{A}' \models (A \leftrightarrow G)$ because $\mathcal{A}'(A) = \mathcal{A}(G)$ and $\mathcal{A}(G) = \mathcal{A}'(G)$ by the Coincidence Lemma (Exercise 1.2).
Thus $F \wedge (A \leftrightarrow G)$ is satsifiable (by $\mathcal{A}'$).
Conversely we actually have $F \wedge (A \leftrightarrow G) \models F[G/A]$.
Suppose $\mathcal{A} \models F \wedge (A \leftrightarrow G)$. This implies $\mathcal{A}(A) = \mathcal{A}(G)$.
Therefore $\mathcal{A}[\mathcal{A}(G)/A] = \mathcal{A}$.
Thus $\mathcal{A}(F[G/A]) = (\mathcal{A}[\mathcal{A}(G)/A])(F) = \mathcal{A}(F) = 1$ by the Substitution Lemma.

Does $F[G/A] \models F \wedge (A \leftrightarrow G)$ hold?

Summary

Theorem
For every formula F of size n there is an equisatisfiable CNF formula G of size $O(n)$.

Similarly it can be shown:

Theorem
For every formula F of size n there is an equivalid DNF formula G of size $O(n)$.

Validity of CNF

Validity of formulas in CNF can be checked in linear time.

A formula in CNF is valid iff all its disjunctions are valid.
A disjunction is valid iff it contains both an atomic A and $\neg A$ as literals.

Example

Valid: $(A \vee \neg A \vee B) \wedge (C \vee \neg C)$

Not valid: $(A \vee \neg A) \wedge (\neg A \vee C)$

Satisfiability of DNF

Satisfiability of formulas in DNF can be checked in linear time.

A formula in DNF is satisfiable iff at least one of its conjunctions is satisfiable. A conjunction is satisfiable iff it does not contain both an atomic A and $\neg A$ as literals.

Example

Satisfiable: $(\neg B \wedge A \wedge B) \vee (\neg A \wedge C)$

Unsatisfiable: $(A \wedge \neg A \wedge B) \vee (C \wedge \neg C)$

Satisfiability/validity of DNF and CNF

Theorem
Satisfiability of formulas in CNF is NP-complete.

Theorem
Validity of formulas in DNF is co-NP-complete.

The standard decision procedure for vailidity of F:

1. Transform $\neg F$ into an equisat. formula G in def. CNF
2. Apply efficient CNF-based SAT solver to G