Logic Exercises

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SS 2022 — Exercise Sheet 5 — 27.05.2022

Exercise 5.1. [A Family of Formulas]
Show that the following schema has a proof in natural deduction for all $n \geq 1$:

$$P_n = ((A_1 \wedge (A_2 \wedge (\cdots \wedge A_n) \cdots) \rightarrow B) \rightarrow (A_1 \rightarrow (A_2 \rightarrow (\cdots (A_n \rightarrow B) \cdots))))$$

Solution:

- Case $n = 1$:
  Formula: $P_1 = (A_1 \rightarrow B) \rightarrow (A_1 \rightarrow B)$
  Proof: trivial using $(\rightarrow I)$
- Case $n + 1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A_2 \wedge \cdots \wedge A_{n+1} \rightarrow B) \rightarrow (A_2 \rightarrow \cdots \rightarrow A_{n+1} \rightarrow B)}\text{(IH)} \qquad \dfrac{\overline{B}\ \text{(subproof)}}{A_2 \wedge \cdots \wedge A_{n+1} \rightarrow B} \rightarrow I_3}{A_2 \rightarrow \cdots \rightarrow A_{n+1} \rightarrow B} \rightarrow E}{A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_{n+1} \rightarrow B} \rightarrow I_2}{(A_1 \wedge A_2 \wedge \cdots \wedge A_{n+1} \rightarrow B) \rightarrow (A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_{n+1} \rightarrow B)} \rightarrow I_1$$

Subproof:

$$\dfrac{[A_1 \wedge A_2 \wedge \cdots \wedge A_{n+1} \rightarrow B]^1 \qquad \dfrac{[A_1]^2 \qquad [A_2 \wedge \cdots \wedge A_{n+1}]^3}{A_1 \wedge A_2 \wedge \cdots \wedge A_{n+1}} \wedge I}{B} \rightarrow E$$

Exercise 5.2. [From Sequent Calculus to Natural Deduction]
How can we construct a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$ from a sequent calculus proof $\Gamma \Rightarrow \Delta$?

Solution:

1. Known from lecture: If $\Gamma \Rightarrow \Delta$ then $\Gamma, \neg\Delta \vdash_{\mathrm{N}} \bot$
2. If $\Gamma, \neg\Delta \vdash_{\mathrm{N}} \bot$ then $\Gamma, \neg \bigvee \Delta \vdash_{\mathrm{N}} \bot$ (to be shown)
3. Use the $(\bot)$ rule to go from $\Gamma, \neg \bigvee \Delta \vdash_{\mathrm{N}} \bot$ to $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$

Proof of 2.: First we show that $\neg \bigvee \Delta \vdash \neg F_i$ for any $F_i \in \Delta$. Let $\neg \bigvee \Delta = \neg(F_1 \vee (\ldots (F_{n-1} \vee F_n) \ldots))$. Then

$$\dfrac{\dfrac{\neg(F_1 \vee (\ldots (F_{n-1} \vee F_n) \ldots)) \qquad \dfrac{[F_i]^1}{F_1 \vee (\ldots (F_{n-1} \vee F_n) \ldots)}(*)}{\bot}(\neg E)}{\neg F_i}(\neg I)_1$$

We prove $(*)$ by induction on n. The case $n = 1$ is trivial. In case $n + 1$, we consider two cases. Case $i = 1$:

$$\dfrac{[F_1]}{F_1 \vee (\ldots (F_n \vee F_{n+1}) \ldots)}(\vee I_1)$$

Case $i > 1$:

$$\dfrac{\dfrac{[F_i]}{F_2 \vee (\ldots (F_n \vee F_{n+1}) \ldots)}(IH)}{F_1 \vee (\ldots (F_n \vee F_{n+1}) \ldots)}(\vee I_2)$$

Now given a proof of $\Gamma, \neg\Delta \vdash_{\mathrm{N}} \bot$, we replace all open assumptions $\neg F_i$ for $F_i \in \Delta$ in the proof by a proof of $\Gamma, \neg \bigvee \Delta \vdash_{\mathrm{N}} \neg F_i$ as constructed in the previous step. This gives us a proof of $\Gamma, \neg \bigvee \Delta \vdash_{\mathrm{N}} \bot$

Exercise 5.3. [Hilbert Calculus]
Prove the following formula with a linear proof in Hilbert calculus: $(F \wedge G) \to (G \wedge F)$

Hint: Use the deduction theorem.

Solution:
We first apply the deduction theorem. It remains to construct a proof of $F \wedge G \vdash_H G \wedge F$.

1.	$F \wedge G \to G$	A5
2.	$F \wedge G$	Γ
3.	$F \wedge G \to F$	A4
4.	$G \to F \to G \wedge F$	A3
5.	G	1, 2
6.	$F \to G \wedge F$	4, 5
7.	F	3, 2
8.	$G \wedge F$	6, 7

Exercise 5.4. [From Hilbert Calculus to Natural Deduction]
Prove: if $\Gamma \vdash_H F$ then $\Gamma \vdash_N F$.

Solution:
A proof tree in $\vdash_H$ consists of repeated applications of the $(\to E)$ rule, where each leaf is closed by one of the axioms $(A_1) - (A_9)$. Rule $(\to E)$ is already an axiom in ND. Each (A_i) is provable in ND (to be shown). The claim then follows by induction on the height of the proof tree in $\vdash_H$.

The proofs of $(A_1) - (A_9)$ in ND are all straightforward. Here is one example:

$$\dfrac{\dfrac{\dfrac{[\neg F \to \bot]_1 \qquad [\neg F]_2}{\bot}\,(\to E)}{F}\,(\bot)_2}{(\neg F \to \bot) \to F}\,(\to I)_1$$

Homework 5.1. [Small Hilbert] (++)

In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient:

A1 $F \to (G \to F)$

A2 $(F \to G \to H) \to (F \to G) \to F \to H$

A10 $(\neg F \to \neg G) \to (G \to F)$

Derive the following statement from the axioms above with the help of $\to_E$:

$$\neg(F \to F) \to G$$

Optional: In fact, Meredith showed that all that is needed is one single axiom:

$$((((A \to B) \to (\neg C \to \neg D)) \to C) \to E) \to ((E \to A) \to (D \to A))$$

Try to derive some axiom of your choice presented in the lecture in Meredith's system.

Solution:

We apply the deduction theorem, which follows from only A1 and A2. We then need to construct a proof of $\neg(F \to F) \vdash_H G$:

1.	$\neg(F \to F)$	Γ
2.	$\neg(F \to F) \to \neg G \to \neg(F \to F)$	A1
3.	$\neg G \to \neg(F \to F)$	$\to_E$ 1. and 2.
4.	$(\neg G \to \neg(F \to F)) \to ((F \to F) \to G)$	A10
5.	$(F \to F) \to G$	$\to_E$ 3. and 4.
6.	$(F \to F)$	$(*)$
7.	G	$\to_E$ 6. and 5.

Here, $(*)$ is the proof of $F \to F$ from the lecture, which only uses A1 and A2:

1.	$F \to ((F \to F) \to F)$	*A*1
2.	$(F \to (F \to F) \to F) \to (F \to F \to F) \to F \to F$	A2
3.	$(F \to F \to F) \to F \to F$	$\to_E$ 1. and 2.
4.	$F \to F \to F$	A1
5.	$F \to F$	$\to_E$ 3. and 4.

Homework 5.2. [From Sequent Calculus to Natural Deduction: Reloaded] (++)
In Exercise 5.2, we constructed a natural deduction proof $\Gamma \vdash_{\mathrm{N}} \bigvee \Delta$ from a sequent calculus proof of $\Gamma \Rightarrow \Delta$. That construction created classical proofs because it required the use of the $(\bot)$ rule.

Let us consider yet another restricted Sequent Calculus called "G3c". In G3c, we have $\Delta = \{F\}$, that is the succedent always contains exactly one formula. Here are the axioms:

$$\text{Ax} \quad P, \Gamma \Rightarrow P \ (P \text{ atomic}) \qquad\qquad \text{L}\bot \quad \bot, \Gamma \Rightarrow A$$

$$\text{L}\wedge \ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\wedge \ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\vee \ \frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\vee \ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1} \ (i = 0, 1)$$

$$\text{L}\!\to \ \frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad\qquad \text{R}\!\to \ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

Give a direct construction that transforms a G3c proof $\Gamma \Rightarrow F$ into a natural deduction proof $\Gamma \vdash_{\mathrm{N}} F$ without using the $(\bot)$ rule. You are, however, allowed to use the intuitionistic rule $\bot \vdash_N F$; call it $(\bot E)$.

Solution:
As usual, by induction on the proof tree + case distinction on final rule application:

Case L⊥: Follows by using $(\bot E)$.

Case L∧:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{C}\,(\text{IH using }1,2)}{A \to C}\,(\to I)_2 \qquad \dfrac{[A \wedge B]}{A}\,(\wedge E_1)}{\dfrac{C}{B \to C}\,(\to I)_1} \qquad \dfrac{[A \wedge B]}{B}\,(\wedge E_2)}{C}\,(\to E)$$

Case R∧: Follows from $(\wedge I) + IH$

Case L∨:

$$\dfrac{[A \vee B] \qquad \dfrac{\overset{[A]}{\vdots}}{C}\,(IH) \qquad \dfrac{\overset{[B]}{\vdots}}{C}\,(IH)}{C}\,(\vee E)$$

Case R∨: $(IH) + (\vee I_i)$, $i \in \{1,2\}$

Case L→:

of second premise ↓ (IH+1) ; of first premise ↓ (IH)

$$\dfrac{\dfrac{\dfrac{}{C}\,(IH+1)}{B \to C}\,(\to I)_1 \qquad \dfrac{[A \to B] \qquad \dfrac{}{A}\,(IH)}{B}\,(\to E)}{C}\,(\to E)$$

Case R→: $(IH) + (\to I)$.

□

Homework 5.3. **[Simulating Truth Tables]** (+)
In the lecture, the following lemma was discussed:

Let atoms$(F) \subseteq \{A_1, \ldots, A_n\}$. Then we can construct a proof $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} F^{\mathcal{A}}$.

Recall the definition of $F^{\mathcal{A}}$:

$$F^{\mathcal{A}} = \begin{cases} F, & \text{if } \mathcal{A}(F) = 1 \\ \neg F, & \text{otherwise} \end{cases}$$

The proof proceded by induction on F. The cases for atomic formulas as well as implication were shown in the lecture. Prove the cases for negation and disjunction!

Solution:
Case $F \equiv \neg F'$: Assume $\mathcal{A}(F) = 1$. Then $\mathcal{A}(F') = 0$. Hence, $F^{\mathcal{A}} = F = \neg F' = F'^{\mathcal{A}}$. By the IH, we have a proof of $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} F'^{\mathcal{A}}$, which finishes the case.

Assume $\mathcal{A}(F) = 0$. Then $\mathcal{A}(F') = 1$. Hence, $F^{\mathcal{A}} = \neg F = \neg\neg F'$ and $F'^{\mathcal{A}} = F'$. By the IH, we have a proof of $A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} F'^{\mathcal{A}}$. To conclude, we build the following proof tree:

$$\dfrac{\dfrac{[\neg F']^1 \qquad \dfrac{}{F'}(IH)}{\bot}(\rightarrow E)}{\neg\neg F'}(\rightarrow I)_1$$

Case $F \vee G$: Assume $\mathcal{A}(F) = 1$. Then $\mathcal{A}(F) = 1$ or $\mathcal{A}(G) = 1$. In case of the former, we build

$$\dfrac{\dfrac{}{F}(IH)}{F \vee G}(\vee I_1)$$

Note that $atoms(F \vee G) = atoms(F) \cup atoms(G)$, which we used in order to apply the IH. The other case is symmetric.

Assume $\mathcal{A}(F \vee G) = 0$. Then $\mathcal{A}(F) = 0$ and $\mathcal{A}(G) = 0$. By the IH, we obtain proofs

$$A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} \neg F$$
$$A_1^{\mathcal{A}}, \ldots, A_n^{\mathcal{A}} \vdash_{\mathrm{N}} \neg G$$

To conclude, we build the following proof tree:

$$
\dfrac{
  \dfrac{
    \dfrac{
      [F \lor G]^1 \qquad
      \dfrac{\dfrac{[\neg F \land \neg G]^2}{\neg F}(\land E_1) \qquad [F]^3}{\bot}(\to E) \qquad
      \dfrac{\vdots}{\bot}(\to E)
    }{\bot}(\lor E)_3
  }{\neg F \land \neg G \to \bot}(\to I)_2
  \qquad
  \dfrac{\dfrac{}{\neg F}(IH) \qquad \dfrac{}{\neg G}(IH)}{\neg F \land \neg G}(\land I)
}{
  \dfrac{\bot}{\neg (F \lor G)}(\to I)_1
}(\to E)
$$

> Simplicity is the ultimate sophistication.
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> — William Gaddis