

# LOGIC EXERCISES

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EXERCISE SHEET 11

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**Exercise 11.1. [This Is Unnatural]**

Let  $\mathcal{N}$  be the standard model of the natural numbers. In tutorial exercise 10.4, we proved that any countable axiomatisation  $T$  of  $\mathcal{N}$  (i.e.  $Cn(T) = Th(\mathcal{N})$ ) over some signature  $\Sigma$  admits another, non-isomorphic model  $\mathcal{N}'$  (in particular,  $Cn(T) = Th(\mathcal{N}')$ ).

Prove that  $\mathcal{N}'$  contains not only one, but infinitely many non-standard natural numbers  $d_n \in U^{\mathcal{N}'}$ , i.e.  $\mathcal{N}'[d_n/x] \models x \neq m$  for all  $n, m \in \mathbb{N}$  and  $\mathcal{N}'[d_n/x, d_m/y] \models x \neq y$  for all  $n, m \in \mathbb{N}$  with  $n \neq m$ .

**Exercise 11.2. [Decidability of Complete Theories]**

Assume  $S$  is finitely axiomatizable and complete, i.e.  $F \in S$  or  $\neg F \in S$  for any sentence  $F$ .

1. Given only the axiomatization of  $S$ , give a procedure deciding whether  $S \models F$  for any sentence  $F$ .
2. Can you obtain a similar result when the assumption is that the axiom system is only *recursively enumerable*?

**Exercise 11.3. [One Finite, All Finite]**

Show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if  $Cn(\Gamma) = Cn(\Delta)$  with  $\Gamma$  countable and  $\Delta$  finite, then there is a finite  $\Gamma' \subseteq \Gamma$  with  $Cn(\Gamma') = Cn(\Gamma)$ . Can you also obtain  $\Gamma'$  effectively?

**Exercise 11.4. [Natural Deduction]**

Prove the following formula using natural deduction.

$$\neg(\forall x(\exists y(\neg P(x) \wedge P(y))))$$

**Homework 11.1. [Counterexamples from Sequent Calculus]** (++)

Consider the statement  $\forall x(P(x) \rightarrow \neg P(f(x)))$ .

1. What happens when trying to prove the validity of this formula in sequent calculus?
2. How can we derive a countermodel from the proof tree?
3. Is there a smaller countermodel?

**Homework 11.2. [Natural Deduction]** (++)

Prove the following statements using natural deduction.

1.  $\neg\forall x\exists y\forall z(\neg P(x, z) \wedge P(z, y))$
2.  $\exists x(P(x) \rightarrow \forall xP(x))$

**Homework 11.3. [Closure Operator]** (+)

Show that  $Cn$  is a **closure operator**, i.e.  $Cn$  fulfills the following properties:

- $S \subseteq Cn(S)$
- if  $S \subseteq S'$  then  $Cn(S) \subseteq Cn(S')$
- $Cn(Cn(S)) = Cn(S)$

**Homework 11.4. [Elementary Classes]** (++)

In this exercise, we assume that all structures and formulas share the same signature  $\Sigma$ .

We define the operator  $Mod(S)$  that returns the class of all structures that model a set of formulas  $S$ . In other words,  $Mod(S)$  contains all  $\mathcal{A}$  such that  $\mathcal{A} \models S$ .

A class of models  $M$  is said to be *elementary* if there is a set of formulas  $S$  such that  $M = Mod(S)$ . If  $S$  is just a singleton set, i.e. there is a formula  $F$  such that  $S = \{F\}$ , then  $M$  is *basic elementary*.

Prove:

1. A class of models  $M$  is basic elementary if and only if there is a *finite* set of formulas  $S$  such that  $M = Mod(S)$ .
2. If  $M$  is basic elementary and  $M = Mod(S)$  for countable  $S$ , then there is a finite subset  $S' \subseteq S$  such that  $M = Mod(S')$ .

The logic of the world is prior to all truth and falsehood.

— Ludwig Wittgenstein<sup>1</sup>

<sup>1</sup>Yes, Ludwig strikes again – he just dropped too many great quotes.