

Arrows

$\xrightarrow{0}$	$:= \{(x, x) \mid x \in A\}$	identity
$\xrightarrow{n+1}$	$:= \xrightarrow{n} \circ \rightarrow$	$(n + 1)$ -fold composition
$\xrightarrow{+}$	$:= \bigcup_{i>0} \xrightarrow{i}$	transitive closure
$\xrightarrow{*}$	$:= \xrightarrow{+} \cup \xrightarrow{0}$	reflexive transitive closure
$\xrightarrow{=}$	$:= \rightarrow \cup \xrightarrow{0}$	reflexive closure
$\xrightarrow{-1}$	$:= \{(y, x) \mid x \rightarrow y\}$	inverse
\leftarrow	$:= \xrightarrow{-1}$	inverse
\leftrightarrow	$:= \rightarrow \cup \leftarrow$	symmetric closure
$\xleftrightarrow{+}$	$:= (\leftrightarrow)^{+}$	transitive symmetric closure
$\xleftrightarrow{*}$	$:= (\leftrightarrow)^{*}$	reflexive transitive symmetric closure

Terminology

- x is **reducible** iff there is a y such that $x \rightarrow y$.
- x is **in normal form (irreducible)** iff it is not reducible.
- y is a **normal form of x** iff $x \xrightarrow{*} y$ and y is in normal form. If x has a uniquely determined normal form, the latter is denoted by $x \downarrow$.
- y is a **direct successor** of x iff $x \rightarrow y$.
- y is a **successor** of x iff $x \xrightarrow{+} y$
- x and y are **joinable (have a common reduct)** iff there is a z such that $x \xrightarrow{*} z \xleftarrow{*} y$, in which case we write $x \downarrow y$.

More terminology

A reduction \rightarrow is called

Church-Rosser iff $x \overset{*}{\leftrightarrow} y \Rightarrow x \downarrow y$

confluent iff $y_1 \overset{*}{\leftarrow} x \overset{*}{\rightarrow} y_2 \Rightarrow y_1 \downarrow y_2$

terminating iff there is no infinite descending chain
 $a_0 \rightarrow a_1 \rightarrow \dots$

normalizing iff every element has a normal form.

convergent iff it is both confluent and terminating.