

Homework is due on May 25, before the tutorial.

Much of this exercise sheet involves the use of Birkhoff's Theorem, which tells us that various relations induced by E are equivalent. So for each problem, we are free to choose the most appropriate relation:

$$s \longleftrightarrow_E^* t \iff E \vdash s \approx t \iff E \models s \approx t$$

Exercise 1 (H) (*Equivalence Classes*)

Let $\Sigma = \{f, g\}$ and

$$E = \{f(f(x)) \approx f(x), g(f(x)) \approx f(x), f(g(x)) \approx g(x), g(g(x)) \approx g(x)\}$$

- Describe the equivalence classes of \approx_E .
- For each equivalence class $[t]_{\approx_E}$, determine a shortest term \hat{t} in $[t]_{\approx_E}$.
- Give a model for E that has more than one element.

Exercise 2 (H) (*Congruence Closure*)

Let $\Sigma = \{f, a, b\}$ and $G = \{f(a, b) \approx a\}$. Using congruence closure, decide whether the terms s and t are equivalent with respect to the equation set G . Use the abstract algorithm on equation sets.

- $s = f(f(a, b), b), \quad t = a$
- $s = f(f(a, b), b), \quad t = b$

Exercise 3 (H) (*More General Substitutions*)

Let σ and σ' be substitutions such that each one is an instance of the other: Formally, we have both $\sigma \lesssim \sigma'$ and $\sigma \gtrsim \sigma'$. Show that in this case, there must exist a renaming ρ (i.e., an injective substitution where $\text{Ran}(\rho) \subseteq V$) such that $\sigma = \rho\sigma'$.

(Continued on back)

Exercise 4 (T) (*Consistency*)

A set E of equations is called *consistent* if there exists a model of E with more than one element. Show the following statements:

- a) E is inconsistent if and only if $E \vdash x \approx y$ (for distinct variables x and y) is derivable as a syntactic consequence of E .
- b) If E includes an equation of the form $t \approx x$ with $x \notin \text{Var}(t)$, then E is inconsistent.
- c) If there is a model of E with two elements, then for each $n \in \mathbb{N}$ there is a model of E with 2^n elements. Hint: Consider pairs.

Exercise 5 (T) (*Commutativity of +*)

Let $\Sigma = \{0, s, +\}$ and $E = \{x + 0 \approx x, x + s(y) \approx s(x) + y\}$.

- a) Show that $E \models x + s(s(0)) \approx s(s(x))$ and $E \models s^i(0) + s^j(0) \approx s^j(0) + s^i(0)$.
- b) Show that $E \not\models x + y \approx y + x$.
Hint: Give a model of E where $+$ is not commutative.