

Exercise 14 (Instances)

Let $T(\Sigma, \{x\})$ be the set of Σ -terms over the single variable x . Moreover let the reduction relation \longrightarrow_I on $T(\Sigma, \{x\})$ be defined as follows: $s \longrightarrow_I t$ iff s is an instance of t , and $s \neq t$.

- Show that \longrightarrow_I is terminating and confluent.
- Does this result generalize to reduction systems with more than one variable?

Exercise 15 (Groups)

Let \circ be a binary, \cdot^{-1} a unary, and e a nullary function symbol. Moreover, assume the following identities:

$$G = \{x \circ (y \circ z) \approx (x \circ y) \circ z, x \circ e \approx x, x \circ x^{-1} \approx e\}$$

Furthermore, you can assume the following derived identity:

$$x^{-1} \circ x \approx_G e$$

- Show $e \circ x \approx_G x$.
- Show $(x \circ y)^{-1} \xrightarrow{*}_G y^{-1} \circ x^{-1}$.

Exercise 16 (Composition of Substitutions)

- Specify $\text{Dom}(\sigma\tau)$ and $\sigma\tau$ for each of the following pairs of substitutions. Use the form $\{x \mapsto \cdot, \dots\}$ for the latter one.
 - $\sigma = \{x \mapsto y\}, \tau = \{y \mapsto x\}$
 - $\sigma = \{x \mapsto y * x\}, \tau = \{x \mapsto y\}$
 - $\sigma = \{x \mapsto y\}, \tau = \{x \mapsto x * y\}$
- Specify conditions on the domains and variable ranges of σ and τ under which $\sigma\tau = \tau$. Show that your conditions are both necessary and sufficient.
- Show that we have $\sigma\tau = \sigma$ if $\forall x \in \text{Dom}(\tau). \sigma(x) = \tau(x)$ and $\text{Dom}(\sigma) \cap \mathcal{VRan}(\tau) = \emptyset$. Is the converse also true in general?

Homework 17 (Composition of Substitutions')

Let σ and τ be substitutions. Describe the variable range $\mathcal{VRan}(\sigma\tau)$ of the composition of σ and τ .

Homework 18 (More Substitutions)

Let σ and σ' be substitutions. Prove or refute:

- a) $\sigma \subseteq \sigma'$ implies $\sigma \lesssim \sigma'$.
- b) $\sigma'\sigma' = \sigma'$ and $\sigma \subseteq \sigma'$ implies $\sigma \lesssim \sigma'$.

Note that in $\sigma \subseteq \sigma'$, a substitution is interpreted as a relation:

$$\sigma \subseteq \sigma' : \iff \text{Dom}(\sigma) \subseteq \text{Dom}(\sigma') \wedge \sigma = \sigma'|_{\text{Dom}(\sigma)}$$

The relation $\sigma \lesssim \sigma'$ tells us that σ is more general than σ' :

$$\sigma \lesssim \sigma' : \iff \exists \delta. \delta\sigma = \sigma'$$

(here δ is a substitution, and $\delta\sigma$ is the usual composition of substitutions)

Homework 19 (Racks)

Every group G can be (conservatively) extended with an additional operation \triangleright , defined as follows:

$$a \triangleright b \approx (a \circ b) \circ a^{-1}$$

Let R denote the extension of G by the above identity, as well as all assumed and proven group identities from exercise 15.

Prove that *self-distributivity* holds for \triangleright :

$$a \triangleright (b \triangleright c) \xrightarrow{*} (a \triangleright b) \triangleright (a \triangleright c)$$

You may only apply one step at a time, except for associativity.