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Equational Logic
Summer Term 2015
Solutions to Exercise Sheet 2

Exercise 8 (Termination)

Show that the following programs, with variables over the natural numbers, terminate on all valid inputs.

a) ggT(m, n)

$$\begin{array}{l}\textbf{while } m \neq n \textbf{ do}\\ \quad \textbf{if } m > n \textbf{ then } m := m - n \textbf{ else } n := n - m\end{array}$$

b) ggT(m, n)

$$\begin{array}{l}\textbf{while } m \neq n \textbf{ do}\\ \quad \textbf{if } m > n \textbf{ then } m := m - n\\ \quad \textbf{else begin } h := m;\ m := n;\ n := h \textbf{ end}\end{array}$$

c) The function f, which is defined recursively as follows.

$$\begin{array}{rcl} f(m, n, 0) & = & m + n \\ f(m, 0, k+1) & = & f(m+k, 1, k) \\ f(m, n+1, k+1) & = & f(m+k, f(m+k, n, k+1), k) \end{array}$$

Solution

a) The program terminates for inputs $n > 0$ and $m > 0$.Due to the loop condition $m \neq n$, we have the loop invariant $n > 0$ und $m > 0$. The program terminates as (m, n) decreases due to the lexicographic order in each iteration of the loop:

$$\begin{array}{lll} (m, n) >_{\mathbb{N}\times\mathbb{N}} (m-n, n) & & \text{since } m > m - n \\ (m, n) >_{\mathbb{N}\times\mathbb{N}} (m, n-m) & & \text{since } n > n - m \end{array}$$

Alternatively, the measure function $\varphi(m, n) = m + n$ works.

b) Similar to a), but with the pair (n, m):

$$\begin{array}{lll} (n, m) >_{\mathbb{N}\times\mathbb{N}} (n, m-n) & & \text{wegen } m > m - n \\ (n, m) >_{\mathbb{N}\times\mathbb{N}} (m, n) & & \text{wegen } n > m \end{array}$$

Alternatively, the measure function $\varphi(m, n) = m + 2 * n$ works.

c) In each recursive call, the triple of parameters decreases due to the *reverse* lexicographic order:

Argument	Argument of recursive call	Reduction
$(m, n, 0)$	no recursive call	
$(m, 0, k+1)$	$(m+k, 1, k)$	$k+1 > k$
$(m, n+1, k+1)$	$(m+k, n, k+1)$	$k+1 = k+1, n+1 > n$
$(n, n+1, k+1)$	$(m+k, f(m+k, n, k+1), k)$	$k+1 > k$

Exercise 9 (Product Order)

Show that the lexicographic product $(A \times B, >_{A\times B})$ of the orders $(A, >_A)$ and $(B, >_B)$ is a total order, if $>_A$ and $>_B$ are total orders.

Solution

Definitions:
$(a, b) >_{A\times B} (a', b') \Leftrightarrow a >_A a' \ \vee \ (a = a' \wedge b >_B b')$
$>$ total $\Leftrightarrow \forall a, b.\ a > b \ \vee \ b > a \ \vee \ a = b$

Goal: $>_A$ total $\wedge >_B$ total $\Rightarrow >_{A\times B}$ total

Proof 1: direct proof.
Let a, a', b, b' be arbitrary but fixed values.
Goal: $(a, b) >_{A\times B} (a', b')$ or $(a', b') >_{A\times B} (a, b)$ or $(a, b) = (a', b')$.

$>_A$ is total, hence: $a >_A a' \ \vee \ a' >_A a \ \vee \ a = a'$.
Case 1: $a >_A a'$ and thus $(a, b) >_{A\times B} (a', b')$.

Case 2: $a' >_A a$ and thus $(a', b') >_{A\times B} (a, b)$.

Case 3: $a = a'$ $(*)$

> As $>_B$ is total, we have: $b >_B b' \ \vee \ b' >_B b \ \vee \ b = b'$.
> Case 3.1: $b >_B b'$ and thus with $(*)$ we get $(a, b) >_{A\times B} (a', b')$.
> Case 3.2: $b' >_B b$ and thus with $(*)$ we get $(a', b') >_{A\times B} (a, b)$.
> Case 3.3: $b = b'$ and thus with $(*)$ we get $(a, b) = (a', b')$.

Proof 2: by contradiction.
Assume $>_{A\times B}$ not total, $>_A$ total, $>_B$ total

$$
\begin{array}{lllll}
>_{A\times B} \text{ not total } \Leftrightarrow \exists (a,b),(a',b'). & \neg & (a,b) & >_{A\times B} & (a',b') & \wedge \\
& \neg & (a',b') & >_{A\times B} & (a,b) & \wedge \\
& \neg & (a,b) = (a',b') & & & \\
\end{array}
$$

$$\neg\ (a, b) = (a', b') \Rightarrow a' \neq a \vee b \neq b'$$

1st Case: $a \neq a'$

$$\left.\begin{array}{lll} \neg\ (a,b) >_{A\times B} (a',b') & \Rightarrow & \neg\ (a >_A a') \\ \neg\ (a',b') >_{A\times B} (a,b) & \Rightarrow & \neg\ (a' >_A a) \end{array}\right\} \Rightarrow\ >_A \text{ not total. Contradiction!}$$

2nd Case: $b \neq b'$ (and $a = a'$)

$$\left.\begin{array}{lllll} \neg & (a,b) >_{A\times B} (a',b') & \Rightarrow & \neg\ (b >_B b') \\ \neg & (a',b') >_{A\times B} (a,b) & \Rightarrow & \neg\ (b' >_B b) \end{array}\right\} \Rightarrow >_B \text{ not total. Contradiction!}$$

Exercise 10 (Positions)

Let s, t be terms, and let $p, q \in \mathrm{Pos}(s)$ be parallel positions in s (i.e., $p \parallel q$). Show $(s[t]_p)|_q = s|_q$.

Solution

By induction on the structure of term s.

Base case: $s = x$ for variable x: By assumption, that p and q are positions in s, we have $p = q = \varepsilon$, which is a contradiction to $p \parallel q$.

Induction step: $s = f(s_1, \ldots, s_n)$. With the same argument as above, we can assume that $p, q \neq \varepsilon$. Hence, let $p = ip'$ and $q = jq'$. We make a case distinction whether $i = j$.

Case $i \neq j$. We assume, w.l.o.g., $i < j$. Then we have

$$(s[t]_p)|_q = f(s_1 \ldots s_i[t]_{p'} \ldots s_j \ldots s_n)|_{jq'} = (s_j)|_{q'} = s|_q$$

Case $i = j$. Then, we have $p' \parallel q'$. By induction hypothesis, we get $(s_i[t]_{p'})|_{q'} = s_i|_{q'}$, and thus $(s[t]_p)|_q = s|_q$. $\square$

Homework 11 (Positions')

Prove the following properties using induction on the length of words denoting positions.

If $p \in \mathrm{Pos}(s)$ and $q \in \mathrm{Pos}(t)$, then

a) $(s[t]_p)|_{pq} = t|_q$

b) $(s[t]_p)[r]_{pq} = s[t[r]_q]_p$

Homework 12 (Playing with the devil)

The devil made up the following game to torture the souls of poor students:

In a box, there is an unknown number of red, green and blue balls. The player has to draw one ball from the box, without seeing its colour beforehand. The drawn ball is removed from the box, but depending on its colour, one of these things happens:

- If it is a red ball, the devil adds an arbitrary number of green balls to the box.
- If it is a green ball, the devil adds an arbitrary number of blue balls to the box.
- If it is a blue ball, nothing happens.

The game ends when the box is empty.

Given any box, what is the probability that the player can end the game in finitely many steps?

Homework 13 (Multiset ordering)

a) Given a strict order $(A, >)$, define the following single-step relation on $\mathcal{M}(A)$:

$$M >_{mul}^1 N :\Leftrightarrow \exists x \in M, Y \in \mathcal{M}(A).\ N = (M - \{x\}) \cup Y \wedge \forall y \in Y.\ x > y$$

Show that the transitive closure of $>_{mul}^1$ is contained in $>_{mul}$.

b) Show that if the ordering $>$ is total, then its multiset extension $>_{mul}$ is total.