

Exercise 14 (Instances)

Let $T(\Sigma, \{x\})$ be the set of Σ -terms over the single variable x . Moreover let the reduction relation \rightarrow_I on $T(\Sigma, \{x\})$ be defined as follows: $s \rightarrow_I t$ iff s is an instance of t , and $s \neq t$.

- a) Show that \rightarrow_I is terminating and confluent.
- b) Does this result generalize to reduction systems with more than one variable?

Solution

Given: $V = \{x\}$ $s \rightarrow_I t \iff s \neq t \wedge \exists \sigma. s = \sigma(t)$

- a) **To show:** \rightarrow_I terminates.

Proof:

Let $\varphi(t) = |\{p | p \in \mathcal{P}os(t) \wedge t|_p \neq x\}|$ “Number of function symbols in t ”
 ($\varphi(t) = |t| - |\mathcal{V}ar(t)|$ also works)

We have:

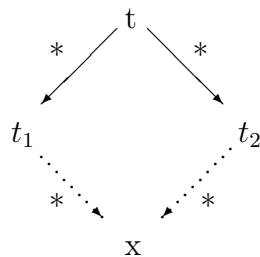
$s \rightarrow_I t \Rightarrow \varphi(s) > \varphi(t)$, as x is the only variable, and, as $s \neq t$, the substitution σ has to map x on a term that contains at least one function symbol.

Hence, φ is a measure function for \rightarrow_I , i.e. \rightarrow_I terminates.

To show: \rightarrow_I is confluent.

Proof:

$\forall t \in T(\Sigma, \{x\}) \setminus \{x\}. t \rightarrow_I x$ (choose $\sigma = \{x \mapsto t\}$)



\Rightarrow , i.e. \rightarrow_I is confluent.

□

b) For confluence: Yes (from a single variable, we can still generate any term).

For termination: No. We have, for example: $f(x, y) \rightarrow_I f(y, x) \rightarrow_I f(x, y) \rightarrow_I \dots$
 i.e. \rightarrow_I does not terminate.

Exercise 15 (Groups)

Let \circ be a binary, \cdot^{-1} a unary, and e a nullary function symbol. Moreover, assume the following identities:

$$G = \{x \circ (y \circ z) \approx (x \circ y) \circ z, x \circ e \approx x, x \circ x^{-1} \approx e\}$$

Furthermore, you can assume the following derived identity:

$$x^{-1} \circ x \approx_G e$$

a) Show $e \circ x \approx_G x$.

b) Show $(x \circ y)^{-1} \xrightarrow{*}_G y^{-1} \circ x^{-1}$.

Solution

We can assume the following identities:

$$\begin{aligned} (1) \quad & x \circ (y \circ z) \approx (x \circ y) \circ z \\ (2) \quad & x \circ e \approx x \\ (3) \quad & x \circ x^{-1} \approx e \\ (5) \quad & x^{-1} \circ x \approx e \end{aligned}$$

a)

$$\frac{\frac{(3) \quad x \circ x^{-1} \approx_G e}{e \approx_G x \circ x^{-1}} \quad \frac{x \approx_G x}{(x \circ x^{-1}) \circ x \approx_G x}}{e \circ x \approx_G (x \circ x^{-1}) \circ x} \quad \frac{\frac{(*) \quad (x \circ x^{-1}) \circ x \approx_G x \circ e}{(x \circ x^{-1}) \circ x \approx_G x} \quad \frac{(2) \quad x \circ e \approx_G x}{x \circ e \approx_G x}}{e \circ x \approx_G x}$$

$$\frac{\frac{(1) \quad x \circ (y \circ z) \approx_G (x \circ y) \circ z}{(x \circ y) \circ z \approx_G x \circ (y \circ z)} \quad \frac{(5) \quad x \approx_G x \quad (x^{-1} \circ x) \approx_G e}{x \circ (x^{-1} \circ x) \approx_G x \circ e}}{(x \circ x^{-1}) \circ x \approx_G x \circ (x^{-1} \circ x)} \quad \frac{x \approx_G x \quad (x^{-1} \circ x) \approx_G e}{x \circ (x^{-1} \circ x) \approx_G x \circ e}}{(*) \quad (x \circ x^{-1}) \circ x \approx_G x \circ (x^{-1} \circ x)}$$

b) Let (4) denote the previously proved identity.

$$(x \circ y)^{-1} \longleftarrow e \circ (x \circ y)^{-1} \quad (4)$$

$$\longleftarrow (y^{-1} \circ y) \circ (x \circ y)^{-1} \quad (5)$$

$$\longleftarrow ((y^{-1} \circ e) \circ y) \circ (x \circ y)^{-1} \quad (2)$$

$$\longleftarrow ((y^{-1} \circ (x^{-1} \circ x)) \circ y) \circ (x \circ y)^{-1} \quad (5)$$

$$\longrightarrow (((y^{-1} \circ x^{-1}) \circ x) \circ y) \circ (x \circ y)^{-1} \quad (1)$$

$$\longleftarrow ((y^{-1} \circ x^{-1}) \circ (x \circ y)) \circ (x \circ y)^{-1} \quad (1)$$

$$\longleftarrow (y^{-1} \circ x^{-1}) \circ ((x \circ y) \circ (x \circ y)^{-1}) \quad (1)$$

$$\longrightarrow (y^{-1} \circ x^{-1}) \circ e \quad (3)$$

$$\longrightarrow y^{-1} \circ x^{-1} \quad (2)$$

Note that because of Birkhoff's Theorem (3.5.14), this suffices to show that $(x \circ y)^{-1} \approx_G y^{-1} \circ x^{-1}$.

Exercise 16 (Composition of Substitutions)

a) Specify $\text{Dom}(\sigma\tau)$ and $\sigma\tau$ for each of the following pairs of substitutions. Use the form $\{x \mapsto \cdot, \dots\}$ for the latter one.

i) $\sigma = \{x \mapsto y\}, \tau = \{y \mapsto x\}$

ii) $\sigma = \{x \mapsto y * x\}, \tau = \{x \mapsto y\}$

iii) $\sigma = \{x \mapsto y\}, \tau = \{x \mapsto x * y\}$

b) Specify conditions on the domains and variable ranges of σ and τ under which $\sigma\tau = \tau$. Show that your conditions are both necessary and sufficient.

c) Show that we have $\sigma\tau = \sigma$ if $\forall x \in \text{Dom}(\tau), \sigma(x) = \tau(x)$ and $\text{Dom}(\sigma) \cap \mathcal{VRan}(\tau) = \emptyset$. Is the converse also true in general?

Solution

a) Definition: $\sigma\tau(x) = \hat{\sigma}(\tau(x))$

i) $\text{Dom}(\sigma\tau) = \{x\}, \sigma\tau = \{x \mapsto y\}$

ii) $\text{Dom}(\sigma\tau) = \{x\}, \sigma\tau = \{x \mapsto y\}$

iii) $\text{Dom}(\sigma\tau) = \{x\}, \sigma\tau = \{x \mapsto y * y\}$

b) $\sigma\tau = \tau$, if $\mathcal{VRan}(\tau) \cap \mathcal{Dom}(\sigma) = \emptyset$ and $\mathcal{Dom}(\sigma) \subseteq \mathcal{Dom}(\tau)$

Proof. Sufficient: Let t be a term. We have $\mathcal{Var}(\tau(t)) \cap \mathcal{Dom}(\sigma) = \emptyset$, as all variables $x \in \mathcal{Dom}(\sigma)$ are also in $\mathcal{Dom}(\tau)$, and thus have been replaced by $\tau(x)$ with $\mathcal{Var}(\tau(x)) \cap \mathcal{Dom}(\sigma) = \emptyset$. Thus, we have $\sigma(\tau(t)) = \tau(t)$.

Necessary: Let $x \in \mathcal{Dom}(\sigma) \setminus \mathcal{Dom}(\tau)$. Then we have $\sigma(\tau(x)) = \sigma(x)$, but $\tau(x) = x$.

Let $x \in \mathcal{VRan}(\tau) \cap \mathcal{Dom}(\sigma)$. Choose y with $x \in \mathcal{Var}(\tau(y))$. Then we have $x \in \mathcal{Var}(\tau(y))$, and thus $\sigma(\tau(y)) \neq \tau(y)$.

c) For a variable $x \in \mathcal{Dom}(\tau)$, we have $\tau(x) = \sigma(x)$ and $\mathcal{Var}(\tau(x)) \notin \mathcal{Dom}(\sigma)$, thus $\sigma(\tau(x)) = \tau(x) = \sigma(x)$. For a variable $x \notin \mathcal{Dom}(\tau)$, we have $\tau(x) = x$, and thus $\sigma(\tau(x)) = \sigma(x)$.

From these base cases, the proposition $\forall t. \sigma(\tau(t)) = \sigma(t)$ follows by straightforward induction on the structure of t .

The converse is not true, as the following counterexample demonstrates: $\sigma = \{x \mapsto y, z \mapsto y\}$ and $\tau = \{x \mapsto z\}$. We have $\tau(x) = z \neq y = \sigma(x)$, but $\sigma\tau = \sigma$.

Homework 17 (Composition of Substitutions')

Let σ and τ be substitutions. Describe the variable range $\mathcal{VRan}(\sigma\tau)$ of the composition of σ and τ .

Homework 18 (More Substitutions)

Let σ and σ' be substitutions. Prove or refute:

- a) $\sigma \subseteq \sigma'$ implies $\sigma \lesssim \sigma'$.
- b) $\sigma'\sigma' = \sigma'$ and $\sigma \subseteq \sigma'$ implies $\sigma \lesssim \sigma'$.

Note that in $\sigma \subseteq \sigma'$, a substitution is interpreted as a relation:

$$\sigma \subseteq \sigma' : \iff \mathcal{Dom}(\sigma) \subseteq \mathcal{Dom}(\sigma') \wedge \sigma = \sigma'|_{\mathcal{Dom}(\sigma)}$$

The relation $\sigma \lesssim \sigma'$ tells us that σ is more general than σ' :

$$\sigma \lesssim \sigma' : \iff \exists \delta. \delta\sigma = \sigma'$$

(here δ is a substitution, and $\delta\sigma$ is the usual composition of substitutions)

Homework 19 (Racks)

Every group G can be (conservatively) extended with an additional operation \triangleright , defined as follows:

$$a \triangleright b \approx (a \circ b) \circ a^{-1}$$

Let R denote the extension of G by the above identity, as well as all assumed and proven group identities from exercise 15.

Prove that *self-distributivity* holds for \triangleright :

$$a \triangleright (b \triangleright c) \xleftarrow{*} \rightarrow_R (a \triangleright b) \triangleright (a \triangleright c)$$

You may only apply one step at a time, except for associativity.