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Equational Logic
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Solutions to Exercise Sheet 5

Exercise 24 (Equivalence Classes)

Let $\Sigma = \{f, g\}$ and

$$E = \{f(f(x)) \approx f(x),\ g(f(x)) \approx f(x),\ f(g(x)) \approx g(x),\ g(g(x)) \approx g(x)\}$$

a) Describe the equivalence classes of $\approx_E$.

b) For each equivalence class $[t]_{\approx_E}$, determine a shortest term $\widehat{t}$ in $[t]_{\approx_E}$.

c) Give a model for E that has more than one element.

Solution

a) We argue by using Birkhoff's theorem.

As Σ only contains unary function symbols, all terms in $T(\Sigma, V)$ are repeated applications of the function symbols f and g to some variable $x \in V$.

An application of an equation in E (reduction relation $\longleftrightarrow_E$) leaves the innermost function symbol unchanged. Thus, we get three equivalence classes for each variable $x \in V$:

i) The set of all terms with $f(x)$ as innermost subterm.

ii) The set of all terms with $g(x)$ as innermost subterm.

iii) The singleton set $\{x\}$.

b) The shortest subterms in the equivalence classes for variable x are $f(x)$, $g(x)$ and x, respectively.

c) We choose a Σ-algebra $\mathcal{A}$ with carrier set $A = \mathbb{N}$ and the following interpretations of the function symbols: $f^{\mathcal{A}} = g^{\mathcal{A}} = id$.

Exercise 25 (Congruence Closure)

Let $\Sigma = \{f, a, b\}$ and $G = \{f(a, b) \approx a\}$. Using congruence closure, decide whether the terms s and t are equivalent with respect to the equation set G. Use the abstract algorithm on equation sets.

a) $s = f(f(a, b), b), \quad t = a$

b) $s = f(f(a, b), b), \quad t = b$

Solution

$$\begin{aligned}
H_0 &= G = \{f(a,b) = a\} \\
H_1 &= \{^{\mathrm{R}}a = a, b = b, f(a,b) = f(a,b), f(f(a,b),b) = f(f(a,b),b), \\
&\qquad {}^{\mathrm{S}}a = f(a,b), {}^{\mathrm{C}}f(f(a,b),b) = f(a,b)\} \\
H_2 &= \{^{\mathrm{S}}f(a,b) = f(f(a,b),b), \\
&\qquad {}^{\mathrm{T}}f(f(a,b),b) = a\}
\end{aligned}$$

At this point, we have shown $f(f(a,b),b) = a$.

$$H_3 = \{^{\mathrm{S}}a = f(f(a,b),b)\}$$

Here, the algorithm terminates and we get $f(f(a,b),b) \neq b$.

Exercise 26 (Unification)

Let u, x, y, and z be variables. Use the unification algorithm recalled in the lecture to solve the following two unification problems:

a) $S_1 = \{f(h(x), g(x,u)) \stackrel{?}{=} f(z, g(f(y,y),z))\}$

b) $S_2 = \{h(x, g(x,y), y) \stackrel{?}{=} h(x, g(a,y), y), z \stackrel{?}{=} h(x, g(x,b), b)\}$

Solution

We proceed by applying one of the four rules Delete, Decompose, Orient and Eliminate in each step.

a)

$$\begin{aligned}
\{f(h(x), g(x,u)) \stackrel{?}{=} f(z, g(f(y,y),z))\} &\implies \textsf{Decompose} \\
\{h(x) \stackrel{?}{=} z, g(x,u) \stackrel{?}{=} g(f(y,y),z)\} &\implies \textsf{Decompose} \\
\{h(x) \stackrel{?}{=} z, x \stackrel{?}{=} f(y,y), u \stackrel{?}{=} z\} &\implies \textsf{Orient} \\
\{z \stackrel{?}{=} h(x), x \stackrel{?}{=} f(y,y), u \stackrel{?}{=} z\} &\implies \textsf{Eliminate} \\
\{z \stackrel{?}{=} h(x), x \stackrel{?}{=} f(y,y), u \stackrel{?}{=} h(x)\} &\implies \textsf{Eliminate} \\
\{z \stackrel{?}{=} h(f(y,y)), x \stackrel{?}{=} f(y,y), u \stackrel{?}{=} h(f(y,y))\} &
\end{aligned}$$

b)

$$
\begin{aligned}
\{h(x, g(x, y), y) \stackrel{?}{=} h(x, g(a, y), y), z \stackrel{?}{=} h(x, g(x, b), b)\} &\Longrightarrow \text{Decompose} \\
\{x \stackrel{?}{=} x, g(x, y) \stackrel{?}{=} g(a, y), y \stackrel{?}{=} y, z \stackrel{?}{=} h(x, g(x, b), b)\} &\Longrightarrow \text{Delete} \\
\{g(x, y) \stackrel{?}{=} g(a, y), z \stackrel{?}{=} h(x, g(x, b), b)\} &\Longrightarrow \text{Decompose} \\
\{x \stackrel{?}{=} a, y \stackrel{?}{=} y, z \stackrel{?}{=} h(x, g(x, b), b)\} &\Longrightarrow \text{Delete} \\
\{x \stackrel{?}{=} a, z \stackrel{?}{=} h(x, g(x, b), b)\} &\Longrightarrow \text{Eliminate} \\
\{x \stackrel{?}{=} a, z \stackrel{?}{=} h(a, g(a, b), b)\} &
\end{aligned}
$$

Homework 27 (Programming Assignment)

You are assigned to implement congruence closure in Haskell. Additional instructions can be found on the lecture web page.

Homework 28 (More General Substitutions)

Let σ and σ' be substitutions such that each one is an instance of the other: Formally, we have both $\sigma \lesssim \sigma'$ and $\sigma \gtrsim \sigma'$. Show that in this case, there must exist a renaming ρ (i.e., an injective substitution where $\mathcal{R}an(\rho) \subseteq V$) such that $\sigma = \rho\sigma'$.