Logics Exercise		
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SS 2016	Exercise Sheet 1	13.04.2016

Submission of Homework: Before tutorial on Apr 20

Exercise 1.1. [Short Questions]
Let M be a set of formulas, and let F and G be formulas. Which of the following assertions hold?

1. If F satisfiable then $M \models F$
2. F is valid iff $\top \models F$
3. If $\models F$ then $M \models F$
4. If $M \models F$ then $M \cup \{G\} \models F$
5. $M \models F$ and $M \models \neg F$ cannot hold simultaneously
6. If $M \models G \rightarrow F$ and $M \models G$ then $M \models F$

Exercise 1.2. [Coincidence Lemma]
Assume that for all atomic formulas A_i in F, $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$. Show that

$$\mathcal{A} \models F \text{ iff } \mathcal{A}' \models F$$

Exercise 1.3. [Semantic Proof]
Let $\models F \rightarrow G$ where F and G do not share any atoms. Show that then either F is unsatisfiable or G is an tautology (or both). *Hint:* you may want to use the previous result.

Exercise 1.4. [Satisfiability Algorithms]
Check the following formulas for satisfiability using one of the algorithms seen in the lecture:

- $F_1 = (\neg A \vee \neg D \vee B) \wedge D \wedge \neg B \wedge E \wedge (\neg D \vee \neg E \vee C)$
- $F_2 = (A \rightarrow C) \wedge (C \rightarrow E) \wedge (\top \rightarrow B) \wedge (C \wedge A \rightarrow D) \wedge (B \rightarrow A) \wedge$
  $(B \rightarrow E) \wedge (D \wedge E \rightarrow \bot)$
- $F_3 = (A \rightarrow E) \wedge (B \rightarrow \bot) \wedge (C \rightarrow B) \wedge (\top \rightarrow A) \wedge (E \wedge B \rightarrow C) \wedge (C \rightarrow D)$

Announcement: You have to achieve a minimum combined score of 50% on your homeworks to be granted a 0.3/0.4 grade bonus on the final exam, provided that the exam is passed.

Homework 1.1. [CNF and DNF] (5 points)
Use the rewriting-based procedure from the lecture to convert the following formulas F and G first to NNF, and then to CNF and DNF. Document each rewriting step.

$$F = \neg\neg\neg(A_1 \wedge \neg(A_2 \vee A_3)) \qquad\qquad G = \neg(A_1 \vee \neg A_2 \wedge (A_3 \vee A_1))$$

Homework 1.2. [Consequence] (5 points)
Let M be a finite set of formulas, and $F_1, \ldots, F_n$ be formulas. Consider the two statements:

1. $\{F_1, \ldots, F_n\} \subseteq M$
2. $M \models \bigwedge_{i=1}^{n} F_i$

Prove that 1) implies 2). Are the statements even equivalent? Proof or counterexample!

Homework 1.3. [Duality and NNF] (5 points)
Let F be a formula in NNF. Let F' be the formula that is constructed from F by replacing every $\wedge$ by $\vee$, every $\vee$ by $\wedge$, every $\neg A_i$ by A_i, and every A_i by $\neg A_i$. Show that, for every assignment $\mathcal{A}$, we have: If $\mathcal{A} \models F$, then $\mathcal{A} \not\models F'$!

Homework 1.4. [Small Parity Formulas] (5 points)
We define the size $|F|$ of a formula F by counting the number of operator symbols and atomic formulas. Formally:

$$|A_i| = 1 \qquad |\neg F| = 1 + |F| \qquad |F \wedge G| = 1 + |F| + |G| \qquad |F \vee G| = 1 + |F| + |G|$$

Your task is to construct a family of formulas $\{F_i \mid i \in \mathbb{N}\}$ such that F_n contains the atomic formulas $A_1, \ldots, A_n$, and we have for every assignment $\mathcal{A}$:

$$\mathcal{A} \models F_n \text{ if and only if } |\{A_i \mid 1 \leq i \leq N \text{ and } \mathcal{A}(A_i) = 1\}| \text{ is odd}$$

Moreover, the size of your formula must be at most quadratic in n, i.e., $|F_n| = O(n^2)$. For simplicity, you may assume that n is a power of two. Justify your construction (both, correctness and size)!