Logics Exercise

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Exercise 9.1. **[β-Function]**
In this exercise, we will consider the β-function. Show that there is a function $\beta : \mathbb{N}^3 \to \mathbb{N}$, such that:

- For every sequence $(a_1, \ldots, a_r) \in \mathbb{N}^r$, there is $t, p \in \mathbb{N}$, such that for all $i \leq r$: $\beta(t, p, i) = a_i$. Intuitively, this means that we can encode every sequence of natural numbers into three natural numbers, and β is the decoding function.
- β can be defined in integer arithmetic, i.e., there is a formula $\phi_\beta(t, p, i, a)$, such that $\mathcal{A} \models \phi_\beta(t, p, i, a)$ iff $\beta(t, p, i) = a$ (Note that we identified semantic numbers and syntactic constants).

Hint: Choose t as a p-adic encoding of the sequence $1, a_1, 2, a_2, \ldots, r, a_r$ for some suitable prime p.

Exercise 9.2. **[Cooper's Algorithm]**
Decide validity of the following formula using Cooper's algorithm.

$$\forall x((2x < 5 \vee 3x < 9) \longrightarrow x < 3)$$

Recall that Cooper's algorithm only converts the formula to NNF, but does not require CNF. Moreover, it introduces a predicate $\not|$ (not divides). Then, the basic statement is: Let F be a formula consisting of disjunctions and conjunctions of atoms of the forms $x < a_i$, $b_i < x$, $\delta_i | x + c_i$, $\varepsilon_i \not| x + d_i$, where a_i, b_i, c_i, d_i are terms not containing x, and δ_i, ε_i are positive integer constants. Let δ be the lcm of all the δ_i, ε_i. Moreover let $F_{-\infty}$ be the formula F, where all upper bounds on x are replaced by $\top$, and all lower bounds on x are replaced by $\bot$. Then:

$$\exists x.F \longleftrightarrow \bigvee_{j=1}^{\delta} F_{-\infty}(j) \vee \bigvee_{j=1}^{\delta} \bigvee_{b_i} F(b_i + j)$$

Exercise 9.3. **[Fake Proof]**
Consider the following proof:

Proposition: Valid first-order formulas are not recursively enumerable.

Proof: Suppose valid formulas were recursively enumerable. Then, we could decide validity of a formula F by enumerating all valid formulas, and stopping when we enumerate F or $\neg F$. As we know that validity of FOL-formulas is undecidable, this yields a contradiction. qed.

What is wrong with the above proof? Are all valid first-order formulas in fact recursively enumerable?

Homework 9.1. **[Undecidability of $Th(\mathbb{Z}, +, \cdot, 0, 1, =)$]** (5 points)
Show that $Th(\mathbb{Z}, +, \cdot, 0, 1, =)$ is undecidable.

Hint: An integer is a natural number iff it is the sum of four integer squares.

Homework 9.2. **[Decidable Axiomatizations]** (7 points)
Let S be a set of sentences of predicate logic.

1. Show: if S has a decidable axiomatization, then S is recursively enumerable.
2. Give a counterexample: if S has a decidable axiomatization, then S is decidable.

Homework 9.3. **[(Un)decidable Problems]** (8 points)
Which of the following problems are decidable? Give your answers considering both, predicate logic with and without equality.

1. Given two formulas of predicate logic, is every structure that is suitable for F and G a model of precisely one of these two formulas?
2. Given a formula F of predicate logic, does F have at least three different models (up to renaming)?
3. Given a formula F of predicate logic, does F have an infinite model? (Warning: the case for prediate logic with equality is substantially more difficult).