TU München
Institut für Informatik

Prof. Tobias Nipkow
Dr. Peter Lammich
Simon Wimmer

SS 2016 | Exercise Sheet 12 | 29.06.2016

Submission of Homework: Before tutorial on July 6

Exercise 12.1. [Resolution and Left-Sided Sequent Calculus]
Consider the left-sided sequent calculus from homework 11.4. Show that every proof tree in the left-sided sequent calculus can be transformed into a resolution proof and vice versa (for propositional logic).

Hint: For the direction from left-sided sequent calculus to resolution you can simplify work by assuming that the sequents only consist of sets of clauses. How can you reduce the number of sequent rules you have to consider in this case?

Exercise 12.2. [From Natural Deduction to Hilbert Calculus]
First prove the following formula with natural deduction:

$$(F \land G) \to (G \land F)$$

Now transform the resulting proof tree to a linear proof or proof tree in Hilbert calculus.

Exercise 12.3. [A Smaller Set of Axioms for Hilbert Calculus]
In the lecture, Hilbert calculus for propositional logic was introduced by means of nine axioms. However, the following three axioms are already sufficient (assuming that $\land$ and $\lor$ are derived connectives):

- A1: $F \to (G \to F)$
- A2: $(F \to G \to H) \to (F \to G) \to F \to H$
- A10: $(\neg F \to \neg G) \to (G \to F)$

(Axioms A1 and A2 already appeared in the lecture). Derive the following axioms from A1, A2 and A10 with help of $\to_E$:

- $\neg(F \to F) \to G$
- $\neg\neg F \to F$

Homework 12.1. [Sequent Calculus for FOL] (5 points)
Prove the following formulas in sequent calculus, or give a countermodel that falsifies the formula.

1. $\neg\exists x P(x) \rightarrow \forall x \neg P(x)$
2. $(\forall x P \vee Q(x)) \rightarrow (P \vee \forall x Q(x))$
3. $\forall x \exists y P(x, y) \rightarrow \exists y \forall x P(x, y)$
4. $\neg(\forall x \exists y \forall z \neg P(x, z) \wedge P(z, y))$

Homework 12.2. [Sequent Calculus (II)] (5 points)
Prove that $\vdash_G \Gamma \Rightarrow \Delta$ implies $\vdash_G \Gamma[t/x] \Rightarrow \Delta[t/x]$, where, for a set of formulas Γ, we define $\Gamma[t/x]$ to be $\{F[t/x] \mid F \in \Gamma\}$, i.e. free occurrences of x are replaced by t. Give two different proofs:

1. A syntactic proof, transforming the proof tree of $\vdash_G \Gamma \Rightarrow \Delta$.
2. A semantic proof, using correctness and completeness of $\vdash_G$.

Homework 12.3. [Marriage Problem] (5 points)
Use the compactness theorem to prove the following: Let B be an infinite set of boys, such that each boy has a finite number of girlfriends. Moreover, any set of k boys has at least k different girlfriends. Show that each boy can marry one of his girlfriends, where polygamy is forbidden.

Hint: Find an (infinite) set of formulas that is satisfiable if and only if a marriage matching exists. Be careful not to form conjunctions or disjunctions over infinite sets!

Homework 12.4. [Size of Resolution Proofs] (5 points)
Find a valid propositional logic formula, such that each sequent calculus proof tree has at least quadratic size in the size of the formula, and the formula has an at most linear size resolution proof.