

LOGICS EXERCISE

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EXERCISE SHEET 1

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Exercise 1.1. [Short Questions]

Let M be a set of formulas, and let F and G be formulas. Which of the following assertions hold?

1. If F satisfiable then $M \models F$
2. F is valid iff $\top \models F$
3. If $\models F$ then $M \models F$
4. If $M \models F$ then $M \cup \{G\} \models F$
5. $M \models F$ and $M \models \neg F$ cannot hold simultaneously
6. If $M \models G \rightarrow F$ and $M \models G$ then $M \models F$

Exercise 1.2. [Coincidence Lemma]

Assume that for all atomic formulas A_i in F , $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$. Show that

$$\mathcal{A} \models F \text{ iff } \mathcal{A}' \models F$$

Exercise 1.3. [Semantic Proof]

Let $\models F \rightarrow G$ where F and G do not share any atoms. Show that then either F is unsatisfiable or G is a tautology (or both). *Hint:* you may want to use the previous result.

Exercise 1.4. [Satisfiability Algorithms]

Check the following formulas for satisfiability using one of the algorithms seen in the lecture:

- $F_1 = (\neg A \vee \neg D \vee B) \wedge D \wedge \neg B \wedge E \wedge (\neg D \vee \neg E \vee C)$
- $F_2 = (A \rightarrow C) \wedge (C \rightarrow E) \wedge (\top \rightarrow B) \wedge (C \wedge A \rightarrow D) \wedge (B \rightarrow A) \wedge (B \rightarrow E) \wedge (D \wedge E \rightarrow \perp)$
- $F_3 = (A \rightarrow E) \wedge (B \rightarrow \perp) \wedge (C \rightarrow B) \wedge (\top \rightarrow A) \wedge (E \wedge B \rightarrow C) \wedge (C \rightarrow D)$

ANNOUNCEMENT: You have to achieve a minimum combined score of 50% on your homeworks to be granted a 0.3/0.4 grade bonus on the final exam, provided that the exam is passed.

Homework 1.1. [CNF and DNF] (5 points)

Use the rewriting-based procedure from the lecture to convert the following formulas F and G first to NNF, and then to CNF and DNF. Document each rewriting step.

$$F = \neg\neg\neg(A_1 \wedge \neg(A_2 \vee A_3)) \qquad G = \neg(A_1 \vee \neg A_2 \wedge (A_3 \vee A_1))$$

Solution: Formula F : Convert to NNF:

$$\begin{aligned} & \neg\neg\neg(A_1 \wedge \neg(A_2 \vee A_3)) \\ \equiv & \neg(A_1 \wedge \neg(A_2 \vee A_3)) \\ \equiv & \neg A_1 \vee \neg\neg(A_2 \vee A_3) \\ \equiv & \neg A_1 \vee (A_2 \vee A_3) \end{aligned}$$

This is already in CNF and DNF!

Formula G : Convert to NNF:

$$\begin{aligned} & \neg(A_1 \vee \neg A_2 \wedge (A_3 \vee A_1)) \\ \equiv & \neg A_1 \wedge \neg(\neg A_2 \wedge (A_3 \vee A_1)) \\ \equiv & \neg A_1 \wedge (\neg\neg A_2 \vee \neg(A_3 \vee A_1)) \\ \equiv & \neg A_1 \wedge (A_2 \vee \neg(A_3 \vee A_1)) \\ \equiv & \neg A_1 \wedge (A_2 \vee (\neg A_3 \wedge \neg A_1)) \end{aligned}$$

To CNF:

$$\begin{aligned} & \neg A_1 \wedge (A_2 \vee (\neg A_3 \wedge \neg A_1)) \\ \equiv & \neg A_1 \wedge (A_2 \vee \neg A_3) \wedge (A_2 \vee \neg A_1) \end{aligned}$$

To DNF:

$$\begin{aligned} & \neg A_1 \wedge (A_2 \vee (\neg A_3 \wedge \neg A_1)) \\ \equiv & (\neg A_1 \wedge A_2) \vee (\neg A_1 \wedge (\neg A_3 \wedge \neg A_1)) \\ \equiv & (\neg A_1 \wedge A_2) \vee (\neg A_3 \wedge \neg A_1) \end{aligned}$$

Homework 1.2. [Consequence] (5 points)

Let M be a finite set of formulas, and F_1, \dots, F_n be formulas. Consider the two statements:

1. $\{F_1, \dots, F_n\} \subseteq M$
2. $M \models \bigwedge_{i=1}^n F_i$

Prove that 1) implies 2). Are the statements even equivalent? Proof or counterexample!

Solution: Assume (1) $\mathcal{A} \models M$. To show $\mathcal{A} \models \bigwedge_{i=1}^n F_i$. With (1): $\mathcal{A} \models F_i$ for all i . With def. of semantics of \wedge : $\mathcal{A} \models \bigwedge_{i=1}^n F_i$. (Formally: Induction on i).

The statements are not equivalent. E.g. $\{A_1 \wedge A_2\} \models A_1 \wedge A_2$.

Homework 1.3. [Duality and NNF] (5 points)

Let F be a formula in NNF. Let F' be the formula that is constructed from F by replacing every \wedge by \vee , every \vee by \wedge , every $\neg A_i$ by A_i , and every A_i by $\neg A_i$. Show that, for every assignment \mathcal{A} , we have: If $\mathcal{A} \models F$, then $\mathcal{A} \not\models F'$!

Solution: By induction on structure of F .

$F = A$ We have $F' = \neg A$. From $\mathcal{A} \models F$ we have $\mathcal{A}(A) = 1$, and thus $\mathcal{A} \not\models F'$.

$F = \neg A$ analogously.

$F = F_1 \wedge F_2$ We have $F' = F'_1 \vee F'_2$. From $\mathcal{A} \models F$ we have $\mathcal{A} \models F_1$ and $\mathcal{A} \models F_2$. By IH, we get $\mathcal{A} \not\models F'_1$ and $\mathcal{A} \not\models F'_2$. Hence, $\mathcal{A} \not\models F'$.

$F = F_1 \vee F_2$ analogously.

qed.

Homework 1.4. [Small Parity Formulas] (5 points)

We define the size $|F|$ of a formula F by counting the number of operator symbols and atomic formulas. Formally:

$$|A_i| = 1 \quad |\neg F| = 1 + |F| \quad |F \wedge G| = 1 + |F| + |G| \quad |F \vee G| = 1 + |F| + |G|$$

Your task is to construct a family of formulas $\{F_i \mid i \in \mathbb{N}\}$ such that F_n contains the atomic formulas A_1, \dots, A_n , and we have for every assignment \mathcal{A} :

$$\mathcal{A} \models F_n \text{ if and only if } |\{A_i \mid 1 \leq i \leq n \text{ and } \mathcal{A}(A_i) = 1\}| \text{ is odd}$$

Moreover, the size of your formula must be at most quadratic in n , i.e., $|F_n| = O(n^2)$. For simplicity, you may assume that n is a power of two. Justify your construction (both, correctness and size)!

Solution: Let's define $F_1 \oplus F_2$ as abbreviation for $(F_1 \wedge \neg F_2) \vee (\neg F_1 \wedge F_2)$ (XOR). We define $F_{k,2^0} = A_{k+1}$ and $F_{k,2^{n+1}} = F_{k,2^n} \oplus F_{k+2^n,2^n}$. Then, we set $F_n = F_{0,n}$.

Moreover, we define $par(j, k) = 1$ iff $|\{A_i \mid j \leq i \leq k \text{ and } \mathcal{A}(A_i) = 1\}|$ is odd and 0 otherwise.

Correctness: We show that $\forall k. \mathcal{A}(F_{k,2^n}) = par(k+1, k+2^n)$, by induction on n :

$$0 \text{ For some arbitrary } k: \mathcal{A}(F_{k,2^0}) = \mathcal{A}(A_{k+1}) = par(k+1, k+1) = par(k+1, k+2^0)$$

$$2^n \longrightarrow 2^{n+1} \text{ For some arbitrary } k:$$

$$\begin{aligned} & \mathcal{A}(F_{k,2^{n+1}}) \\ &= \mathcal{A}(F_{k,2^n} \oplus F_{k+2^n,2^n}) && \text{def of } F_{k,2^n} \\ &= \mathcal{A}(F_{k,2^n}) + \mathcal{A}(F_{k+2^n,2^n}) \pmod 2 && * \\ &= par(k, k+2^n) + par(k+2^n+1, k+2^n+2^n) \pmod 2 && \text{IH} \\ &= par(k, k+2^{n+1}) && ** \end{aligned}$$

where (*) follows e.g. by arguing via the truth table of \oplus and (**) follows from the definition of par .

Size: We have $|F_{\cdot,0}| = 1$ and $|F_{\cdot,n}| = 4 * |F_{\cdot,n/2}| + 5$. Note that the size does not depend on the first parameter. Using the Master-theorem, we immediately get $|F_{\cdot,n}| = O(n^{\log_2 4}) = O(n^2)$.

Alternatively, let $n = 2^k$ for some k . Observe the following expansion:

$$\begin{aligned} |F_{\cdot,n}| &= 4 * |F_{\cdot,n/2}| + 5 \\ &= 4 * (4 * |F_{\cdot,n/4}| + 5) + 5 \\ &= 4^2 * |F_{\cdot,n/4}| + 4^1 * 5 + 4^0 * 5 \\ &= 4^k * |F_{\cdot,0}| + 4^{k-1} * 5 + 4^{k-2} * 5 + \dots + 4^0 * 5 \\ &= 4^k + 5 * \sum_{i=0}^{k-1} 4^i \\ &= 4^k + 5 * \frac{4^k - 1}{4 - 1} \\ &= O(4^k) \\ &= O(4^{\log_2 n}) \\ &= O(n^2) \end{aligned}$$