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SS 2016 | Exercise Sheet 2 | 20.04.2016

**Submission of Homework:** Before tutorial on Apr 27

**Exercise 2.1. [Predicate Logic]**
a) Specify a satisfiable formula $F$, such that for all models $\mathcal{A}$ of $F$, we have $|U_{\mathcal{A}}| \geq 3$.
b) Can you also specify a satisfiable formula $F$, such that for all models $\mathcal{A}$ of $F$, we have $|U_{\mathcal{A}}| \leq 3$?

**Exercise 2.2. [Resolution Completeness]**
a) Does $F \models C$ imply $F \vdash_{Res} C$? Proof or counterexample!
b) Can you prove $F \models C$ by resolution?

**Exercise 2.3. [Resolution of Horn-Clauses]**
Can the resolvent of two Horn-clauses be a non-Horn clause?

**Exercise 2.4. [Optimizing Resolution]**
We call a clause $C$ *trivially true* if $A_i \in C$ and $\neg A_i \in C$ for some atom $A_i$. Show that the resolution algorithm remains complete if it does not consider trivially true clauses for resolution.

**Exercise 2.5. [Finite Axiomatization]**
Let $M_0$ and $M$ be sets of formulas. $M_0$ is called *axiom schema* for $M$, iff for all assignments $\mathcal{A}$: $\mathcal{A} \models M_0$ iff $\mathcal{A} \models M$.

A set $M$ is called *finitely axiomatized* iff there is a finite axiom scheme for $M$.

a) Are all sets of formulas finitely axiomatized? Proof or counterexample? b) Let $M = (F_i)_{i \in \mathbb{N}}$ be a set of formulas, such that for all $i$: $F_{i+1} \models F_i$, and not $F_{i+1} \models F_i$. Is $M$ finitely axiomatized?

**Homework 2.1.** **[Definitonal CNF]** (3 points)
Calculate the definitional CNF of the following formula:

$$(A_1 \vee (A_2 \wedge \neg A_3)) \vee A_4$$

**Solution:**

$$\begin{aligned}
(A_1 \vee (A_2 \wedge \neg A_3)) \vee A_4 \\
\rightsquigarrow \\
((A_1 \vee (A_2 \wedge A_5)) \vee A_4) \wedge (A_5 \leftrightarrow \neg A_3) \\
\rightsquigarrow \\
((A_1 \vee A_6) \vee A_4) \wedge (A_5 \leftrightarrow \neg A_3) \wedge (A_6 \leftrightarrow (A_2 \wedge A_5)) \\
\rightsquigarrow \\
(A_1 \vee A_6 \vee A_4) \wedge CNF(A_5 \leftrightarrow \neg A_3) \wedge CNF(A_6 \leftrightarrow (A_2 \wedge A_5)) \\
\rightsquigarrow (*) \\
(A_1 \vee A_6 \vee A_4) \wedge (A_5 \vee A_3) \wedge (\neg A_5 \vee \neg A_3) \\
\wedge (A_6 \vee \neg A_2 \vee \neg A_5) \wedge (\neg A_6 \vee A_2) \wedge (\neg A_6 \vee A_5)
\end{aligned}$$

(*): By e.g. the truth table approach we get that the CNF of a formula $L_i \leftrightarrow (L_j \wedge L_k)$ is $(L_i \vee \overline{L_j} \vee \overline{L_k}) \wedge (\overline{L_i} \vee L_j) \wedge (\overline{L_i} \vee L_k)$ (where $L_i$, $L_j$ and $L_k$ are literals).

**Homework 2.2. [Definitional DNF]** (5 points)
We call formulas $F$ and $F'$ *equivalid* if

$$\models F \text{ iff } \models F'$$

First show that

$$F[G/A] \text{ and } (A \leftrightarrow G) \rightarrow F \text{ are equivalid}$$

for any formulas $F$ and $G$ and any atom $A$, provided that $A$ does not occur in $G$. Now argue that for every formula $F$ of size $n$ there is an equivalid DNF formula $G$ of size $O(n)$.

**Solution:** Suppose $\mathcal{A} \models F[G/A]$ for any $\mathcal{A}$. We have to consider two cases:

1. $\mathcal{A}(A) = \mathcal{A}(G)$. Then $\mathcal{A}(F[G/A]) = \mathcal{A}(F) = 1$ with the same argument as in the lecture (correctness proof for definitional CNF).
2. $\mathcal{A}(A) \neq \mathcal{A}(G)$. Then $\mathcal{A} \not\models A \leftrightarrow G$.

In either case we immediately get $\mathcal{A} \models (A \leftrightarrow G) \rightarrow F$, which completes the 'only if'-direction.

For the other direction, assume $\models (A \leftrightarrow G) \rightarrow F$ and let $\mathcal{A}$ be an suitable assignment for $F[G/A]$. We obtain an assingment $\mathcal{A}'$, which is not defined for $A$ (it may be the case that $\mathcal{A} = \mathcal{A}'$). By the coincidence lemma, we know $\mathcal{A} \models F[G/A]$ iff $\mathcal{A}' \models F[G/A]$. We now extend $\mathcal{A}'$ to some assignment $\mathcal{A}'' = \mathcal{A}' \cup \{A \mapsto \mathcal{A}'(G)\}$, where $\mathcal{A}' \models F[G/A]$ iff $\mathcal{A}'' \models F[G/A]$ by the coincidence lemma. We get $\mathcal{A}'' \models (A \leftrightarrow G) \rightarrow F$ by assumption and thus $\mathcal{A}'' \models F$ from the construction of $\mathcal{A}''$. By the substitution lemma, $\mathcal{A}''(F[G/A]) = (\mathcal{A}''[\mathcal{A}''(G)/A])(F) = \mathcal{A}''(F)$. Together we have $\mathcal{A}'' \models F[G/A]$ and thus $\mathcal{A} \models F[G/A]$.

Construction of the definitional DNF now proceeds analogously to the definitional CNF, with the exception that new definitions of the form $A \leftrightarrow G$ are conjoined to the formula via implication and not conjunction. The last step converts the whole formula to DNF. Because $(A \leftrightarrow G) \rightarrow F \equiv (A \wedge \neg G) \vee (\neg A \wedge G) \vee F$ the size of the obtained formula is linear in the size of the original formula.

**Homework 2.3. [Compactness Theorem]** (5 points)
Suppose every subset of $S$ is satisfiable. Show that then

every subset of $S \cup \{F\}$ is satisfiable or
every subset of $S \cup \{\neg F\}$ is satisfiable

for any formula $F$.

**Solution:** Proof by contradiction. Suppose $S \cup \{F\}$ has an unsatisfiable subset $M$ and $S \cup \{\neg F\}$ has an unsatisfiable subset $L$. We can assume that $M = M' \cup \{F\}$ and $L = L' \cup \{\neg F\}$ for some $M'$, $L'$ where $M' \subseteq S$ and $L' \subseteq S$ because every subset of $S$ is satisfiable. We additionally know that $M' \cup L'$ is satisfiable by assumption. Consider the sets

$$M' \cup L' \cup \{F\} \qquad \text{and} \qquad M' \cup L' \cup \{\neg F\}$$

Then one of them has to be satisfiable. (Let $\mathcal{A}$ with $\mathcal{A} \models M' \cup L'$. Then either $\mathcal{A} \models F$ or $\mathcal{A} \not\models F$. That is, $\mathcal{A} \models F$ or $\mathcal{A} \models \neg F$.) This directly implies that either $M$ or $L$ is satisfiable, a contradiction.

**Homework 2.4.** **[Compactness and Validity]** (2 points)
We say that a set of formulas $S$ is valid if every $F$ in $S$ is valid. Prove or disprove:

$$S \text{ is valid iff every finite subset of } S \text{ is valid}$$

**Solution:** Trivial by definition:

$$\begin{array}{lll} S \text{ valid} & \text{iff} & \text{all } F \text{ in } S \text{ are valid} \\ & \text{iff} & \text{all finite subsets of } S \text{ are valid.} \end{array}$$

**Homework 2.5.** **[Resolution]** (5 points)
Use the resolution procedure to decide if the following formulas are satisfiable. Show your work (by giving the corresponding DAG or linear derivation)!

1. $\neg A_1 \wedge A_2 \wedge (\neg A_1 \vee A_3) \wedge (A_1 \vee \neg A_2 \vee A_3)$
2. $A_2 \wedge (\neg A_3 \vee A_1) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1) \wedge (\neg A_2 \vee A_3)$

**Solution:**

1. 0: $\{\neg A_1\}$
   1: $\{A_2\}$
   2: $\{\neg A_1, A_3\}$
   3: $\{A_1, \neg A_2, A_3\}$
   4: $\{\neg A_2, A_3\}$ (0, 3)
   5: $\{A_1, A_3\}$ (1, 3)
   6: $\{A_3\}$ (0, 5)
   No more inferences are possible and thus we conclude that the formula is satisfiable.

2. 0: $\{A_2\}$
   1: $\{\neg A_3, A_1\}$
   2: $\{\neg A_1, A_2\}$
   3: $\{\neg A_1\}$
   4: $\{\neg A_2, A_3\}$
   5: $\{\neg A_1\}$ (0, 2)
   6: $\{A_3\}$ (0, 4)
   7: $\{\neg A_3\}$ (1, 3)
   8: $\square$ (6, 7)
   The formula is unsatisfiable.