Logics Exercise

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Homework 3.1. [Equivalence] (4 points)
Let F and G be arbitrary formulas. (In particular, they may contain free occurrences of x.) Which of the following equivalences hold? Proof or counterexample!

1. $\forall x(F \wedge G) \equiv \forall xF \wedge \forall xG$
2. $\exists x(F \wedge G) \equiv \exists xF \wedge \exists xG$

Solution: 1) holds. Assume $\mathcal{A} \models \forall x(F \wedge G)$,
$\iff$ for all $d \in U_{\mathcal{A}}$, we have $\mathcal{A}[d/x] \models F$ and $\mathcal{A}[d/x] \models G$,
$\iff$ for all $d_1 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_1/x] \models F$ and for all $d_2 \in U_{\mathcal{A}}$, we have $\mathcal{A}[d_2/x] \models G$
$\iff$ $\mathcal{A} \models \forall xF \wedge \forall xG$

2) does not hold. Let $F = P(x)$ and $G = Q(x)$, $U_{\mathcal{A}} = \{0, 1\}$, $P^{\mathcal{A}} = \{0\}$, and $Q^{\mathcal{A}} = \{1\}$. Clearly, $\mathcal{A} \models \exists xF \wedge \exists xG$ but $\mathcal{A} \not\models \exists x(F \wedge G)$

Homework 3.2. [Preorders] (4 points)
A reflexive and transitive relation is called *preorder*. In predicate logic, preorders can be characterized by the formula

$$F \equiv \forall x \forall y \forall z\ (P(x,x) \wedge (P(x,y) \wedge P(y,z) \longrightarrow P(x,z)))$$

Which of the following structures are models of F? No proofs are required for the positive case. Give counterexamples for the negative case!

1. $U^{\mathcal{A}} = \mathbb{N}$ and $P^{\mathcal{A}} = \{(m,n) \mid m = n\}$
2. $U^{\mathcal{A}} = 2^{\mathbb{N}}$ and $P^{\mathcal{A}} = \{(A,B) \mid A \supseteq B\}$
3. $U^{\mathcal{A}} = \mathbb{Z}$ and $P^{\mathcal{A}} = \{(x,y) \mid\ 5 > |x-y|\ \}$

Solution: 1,2 are obviously preorders.

3. This is not transitive, e.g., $5 > |1-3|$ and $5 > |3-6|$, but $5 \not> |1-6|$

Homework 3.3. **[Infinite Models]** (5 points)
Consider predicate logic with equality. We use infix notation for equality, and abbreviate $\neg(s = t)$ by $s \neq t$. Moreover, we call a structure finite iff its universe is finite.

1. Specify a finite model for the formula $\forall x\ (c \neq f(x) \wedge x \neq f(x))$.
2. Specify a model for the formula $\forall x \forall y\ (c \neq f(x) \wedge (f(x) = f(y) \longrightarrow x = y))$.
3. Show that the above formula has no finite model.

Solution:

1. $U^{\mathcal{A}} = \{0, 1, 2\} \subset \mathbb{N}$ and $c^{\mathcal{A}} = 0$ and $f^{\mathcal{A}}(0) = 1 \mid f^{\mathcal{A}}(n+1) = 2 - n$
2. $U^{\mathcal{A}} = \mathbb{N}$ and $c^{\mathcal{A}} = 0$ and $f^{\mathcal{A}}(n) = n + 1$
3. Assume a model $\mathcal{A}$. First note that the properties transfer to the semantic level, i.e., we have for all $x, y \in U_{\mathcal{A}}$:

$$c^{\mathcal{A}} \neq f^{\mathcal{A}}(x) \tag{1}$$

$$f^{\mathcal{A}}(x) = f^{\mathcal{A}}(y) \implies x = y \tag{2}$$

   Now, we are in a position to show that $U_{\mathcal{A}}$ is infinite. For this, we define $x_i = (f^{\mathcal{A}})^i(c^{\mathcal{A}})$, i.e. i times $f^{\mathcal{A}}$ applied to $c^{\mathcal{A}}$. Clearly, we have $x_i \in U_{\mathcal{A}}$ for all i. We now show that $i < j$ implies $x_i \neq x_j$, immediately yielding infinity of $U_{\mathcal{A}}$. We do induction on i. For 0, we have $x_0 = c^{\mathcal{A}} \neq f^{\mathcal{A}}(\ldots) = x_j$ (by (1)). For $i + 1$, the induction hypothesis gives us $x_i \neq x_j$, which implies $x_{i+1} \neq x_{j+1}$ (by (2)). qed.

Homework 3.4. **[Normal Forms]** (3 points)
Convert the following formula to Skolem form:

$$P(x) \wedge \forall x\ (Q(x) \wedge \forall x \exists y\ P(f(x, y))$$

Show at least the main intermediate conversion stages.

Solution:

$$\begin{aligned}
& P(x) \wedge \forall x(Q(x) \wedge \forall x \exists y P(f(x,y))) && \\
\rightsquigarrow\ & P(x) \wedge \forall x_1(Q(x_1) \wedge \forall x_2 \exists y P(f(x_2,y))) && \text{rectified} \\
\rightsquigarrow\ & \exists x P(x) \wedge \forall x_1(Q(x_1) \wedge \forall x_2 \exists y P(f(x_2,y))) && \text{rectified and closed} \\
\rightsquigarrow\ & \exists x \forall x_1 \forall x_2 \exists y (P(x) \wedge (Q(x_1) \wedge P(f(x_2,y)))) && \text{RPF} \\
\rightsquigarrow\ & \forall x_1 \forall x_2 (P(g) \wedge (Q(x_1) \wedge P(f(x_2, h(x_1,x_2))))) && \text{Skolem form}
\end{aligned}$$

Homework 3.5. **[Relation to Propositional Logic]** (4 points)
Suppose that formula F does not contain any variables or quantifiers. Your task is to construct a *propositional* formula G such that F is valid iff G is valid. Proof that your construction does indeed fulfill this property. Is it also the case that F is satisfiable iff G is satisfiable?

Hints: The approach should define a new *atom* for every *atomic formula* in F. To construct a structure for F from an assignment for G, it may be helpful to use as your universe the set of all terms which can be constructed from function symbols in F. You can assume that F contains at least one constant to ensure that this universe is non-empty.

Solution: G is constructed from F by defining a new atom $A_{P(t_1,\ldots,t_k)}$ for every atomic formula $P(t_1, ..., t_k)$ of G and then recursing over the formula structure of F. For instance if $F = (P(c) \wedge \neg Q(a,b)) \vee Q(b,c)$, then $(A_{P(c)} \wedge \neg A_{Q(a,b)}) \vee A_{Q(b,c)}$.

We need to construct structures for F from assignments for G and vice versa.

(a) Let $\mathcal{A}$ be an assignment for G. Let $U_{\mathcal{A}'}$ be the set of all terms which can be constructed from parts of F. Define $I'_{\mathcal{A}}$ such that

- $I_{\mathcal{A}'}(f(t_1, ..., t_k)) = f(t_1, ..., t_k)$ for any function symbol f and terms $t_1, ...t_k$
- $I_{\mathcal{A}'}(P(t_1, ..., t_k)) = \mathcal{A}(A_{P(t_1,\ldots,t_k)})$ for any predicate symbol P and terms $t_1, ...t_k$

It is easy to show that $I_{\mathcal{A}'}(P(t_1, ..., t_k)) = \mathcal{A}(A_{P(t_1,\ldots,t_k)})$ by induction over the term structure. With induction over the formula structure of F it follows that $I_{\mathcal{A}'}(F) = \mathcal{A}(G)$.

(b) Let $\mathcal{A}' = (U'_{\mathcal{A}}, I'_{\mathcal{A}})$ be a structure of G. Define $\mathcal{A}(A_{P(t_1,\ldots,t_k)}) = I_{\mathcal{A}'}(P(t_1, ..., t_k))$ for any atom of G. It follows via induction over the formula structure of F that $\mathcal{A}(G) = I_{\mathcal{A}'}(F)$.

Now suppose F is valid. Let $\mathcal{A}$ be any assignment for G. By (a) we know that we can construct a structure $\mathcal{A}'$ for F such that $I'_{\mathcal{A}}(F) = \mathcal{A}(G)$. Because F is valid we have $I'_{\mathcal{A}}(F) = \mathcal{A}(G) = 1$. Thus G is valid. An anologous argument using (b) shows that F is valid if G is valid.

Finally, the constructions of (a) and (b) can simiarly easily be used to argue that F is satisfiable iff G is satisfiable.