Logics Exercise

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Exercise 9.1. **[β-Function]**
In this exercise, we will consider the β-function. Show that there is a function $\beta : \mathbb{N}^3 \to \mathbb{N}$, such that:

- For every sequence $(a_1, \ldots, a_r) \in \mathbb{N}^r$, there is $t, p \in \mathbb{N}$, such that for all $i \leq r$: $\beta(t, p, i) = a_i$. Intuitively, this means that we can encode every sequence of natural numbers into three natural numbers, and β is the decoding function.
- β can be defined in integer arithmetic, i.e., there is a formula $\phi_\beta(t, p, i, a)$, such that $\mathcal{A} \models \phi_\beta(t, p, i, a)$ iff $\beta(t, p, i) = a$ (Note that we identified semantic numbers and syntactic constants).

Hint: Choose t as a p-adic encoding of the sequence $1, a_1, 2, a_2, \ldots, r, a_r$ for some suitable prime p.

Exercise 9.2. **[Cooper's Algorithm]**
Decide validity of the following formula using Cooper's algorithm.

$$\forall x((2x < 5 \vee 3x < 9) \longrightarrow x < 3)$$

Recall that Cooper's algorithm only converts the formula to NNF, but does not require CNF. Moreover, it introduces a predicate $\not|$ (not divides). Then, the basic statement is: Let F be a formula consisting of disjunctions and conjunctions of atoms of the forms $x < a_i$, $b_i < x$, $\delta_i | x + c_i$, $\varepsilon_i \not| x + d_i$, where a_i, b_i, c_i, d_i are terms not containing x, and δ_i, ε_i are positive integer constants. Let δ be the lcm of all the δ_i, ε_i. Moreover let $F_{-\infty}$ be the formula F, where all upper bounds on x are replaced by $\top$, and all lower bounds on x are replaced by $\bot$. Then:

$$\exists x.F \longleftrightarrow \bigvee_{j=1}^{\delta} F_{-\infty}(j) \vee \bigvee_{j=1}^{\delta} \bigvee_{b_i} F(b_i + j)$$

Exercise 9.3. **[Fake Proof]**
Consider the following proof:

Proposition: Valid first-order formulas are not recursively enumerable.

Proof: Suppose valid formulas were recursively enumerable. Then, we could decide validity of a formula F by enumerating all valid formulas, and stopping when we enumerate F or $\neg F$. As we know that validity of FOL-formulas is undecidable, this yields a contradiction. qed.

What is wrong with the above proof? Are all valid first-order formulas in fact recursively enumerable?

Homework 9.1. **[Undecidability of $Th(\mathbb{Z}, +, \cdot, 0, 1, =)$]** (5 points)
Show that $Th(\mathbb{Z}, +, \cdot, 0, 1, =)$ is undecidable.

Hint: An integer is a natural number iff it is the sum of four integer squares.

Solution: Let F be a $(\mathbb{N}, +, \cdot, 0, 1, =)$-sentence. We modify F by replacing every subformula of the form $\forall x.G$ by

$$\forall x((\exists n_1 \exists n_2 \exists n_3 \exists n_4\ (x = n_1 * n_1 + n_2 * n_2 + n_3 * n_3 + n_4 * n_4)) \rightarrow G)$$

, and every subformula of the form $\exists x.G$ by

$$\exists x \exists n_1 \exists n_2 \exists n_3 \exists n_4\ (x = n_1 * n_1 + n_2 * n_2 + n_3 * n_3 + n_4 * n_4 \wedge G)$$

, where n_1, n_2, n_3, and n_4 do not appear in G, from the inside out. Let the resulting formula be F'. It follows that $Th(\mathbb{Z}, +, \cdot, 0, 1, =) \models F$ iff $Th(\mathbb{N}, +, \cdot, 0, 1, =) \models F'$ by induction over the structure of F. Thus we have established that $Th(\mathbb{Z}, +, \cdot, 0, 1, =)$ is undecidable by reduction from the decision problem for $Th(\mathbb{N}, +, \cdot, 0, 1, =)$.

Homework 9.2. **[Decidable Axiomatizations]** (7 points)
Let S be a set of sentences of predicate logic.

1. Show: if S has a decidable axiomatization, then S is recursively enumerable.
2. Give a counterexample: if S has a decidable axiomatization, then S is decidable.

Solution: (1) Let A be the decidable set of sentences with $A \models F$ for each $F \in S$. We enumerate all pairs (M, F) of finite subsets $M \subseteq A$ and sentences F. We can eumerate such M by systematically enumerating all sets of sentences and using the decision procedure to check if all generated sentences are a member of A. For each new pair (M, F), we start a resolution procedure to refute $M \models \neg F$, and execute it in parallel to all resolution procedures running already and the enumeration procedure itself (where "parallel" actually means interleaving of processes). Whenever one of these procedures answers "unsat", the corresponding F is the next element in our enumeration. (Note that elements can be duplicated, which does not interfere with enumerability). By compactness, we know that $A \models F$ iff there is a finite $M \subseteq A$ such that $\bigwedge M \wedge \neg F$ is unsat. Thus we know that each $F \in S$ will be enumerated at some point.

(2) Let S be the set of all valid sentences of predicate logic. S has the trivial axiomatization $\emptyset$ but is undecidable.

Homework 9.3. [(Un)decidable Problems] (8 points)
Which of the following problems are decidable? Give your answers considering both, predicate logic with and without equality.

1. Given two formulas of predicate logic, is every structure that is suitable for F and G a model of precisely one of these two formulas?
2. Given a formula F of predicate logic, does F have at least three different models (up to renaming)?
3. Given a formula F of predicate logic, does F have an infinite model? (Warning: the case for prediate logic with equality is substantially more difficult).

Solution:

1. For any formula F' for predicate logic, we let $F := F'$ and $G := \perp$. We know that G doesn't have a model, so if we could decide wether every structure is a model of precisely one of the two formulas, we could decide the validity of F. As we have learned in the lecture, this question is undecidable for both cases.
2. We have seen that every finite model of a formula in prediate logic without equality can be extended to a model of arbitrarily greater size. This also implies that any formula of prediate logic without equality has infinitely many models. Thus for the case without equality the question of asking wether F has more than three models is equivalent to the question of satisfiability of F, and hence undecidable. Every formula of predicate logic without equality is still a formula of predicate logic with equality, and hence the question of deciding wether a formula from predicate logic without equality has more than three models could be reduced to the same question for formulas of predicate logic with equality.
3. Given a formula of predicate logic with equality, assume that we could decide the question wether F has an infinite model. If the answer is yes, we know that F has a model. If the answer is no, we know by the 'extension' argument mentioned above that F cannot have *any* model. For if F had a finite model, then we could extend it to an infinite model but we already know that F does not have an infinite model. Hence we can reduce the question of deciding wether F has a model to deciding wether F has an infinite model, yielding undecidability of the second question.

   For the case of predicate logic with equality, it again holds that a formula of predicate logic without equality is one of predicate logic with equality, and thus the question cannot be decidable for this case as well. Apologies for our confusion on this part.