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SS 2016 Exercise Sheet 11 22.06.2016

**Submission of Homework:** Before tutorial on June 29

**Exercise 11.1.** **[Natural Deduction (Warmup)]**
Prove by natural deduction:

1. $(F \wedge G) \wedge H \rightarrow F \wedge (G \wedge H)$
2. $(F \vee G) \vee H \rightarrow F \vee (G \vee H)$

**Exercise 11.2.** **[Natural Deduction (Advanced)]**
Prove by natural deduction:

1. $\neg(F \wedge G) \rightarrow (\neg F \vee \neg G)$
2. $((F \rightarrow G) \rightarrow F) \rightarrow F$

**Exercise 11.3.** **[Alternative $\wedge$E rule]**
Show how to transform a natural deduction proof that additionally uses the following rule to one that does not use the rule:

$$\frac{F \wedge G \qquad \begin{array}{c}[F, G] \\ \vdots \\ H\end{array}}{H}$$

**Homework 11.1.** **[Natural Deduction (Warmup)]** (3 points)

Show:

$$\vdash_N (F \to G) \to (\neg G \to \neg F)$$

**Solution:**

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg G]^1 \qquad \dfrac{[F]^2 \quad [F \to G]^3}{G} \to E}{\bot} \neg E}{\neg F} \neg I\ (2)}{\neg G \to \neg F} \to I\ (1)}{(F \to G) \to (\neg G \to \neg F)} \to I\ (3)
$$

**Homework 11.2.** **[Classical Reasoning (1)]** (6 points)
We replace rule $\perp$ of the calculus of natural deduction by either one of the following rules:

- $\dfrac{}{F \vee \neg F}$ (law of excluded middle)
- $\dfrac{\neg\neg F}{F}$ (double negation elimination)

Additionally, we add the rule $\dfrac{\perp}{F}$ $(\perp E)$. Show that the calculus of natural deduction remains complete in both cases.

**Solution:** We want to show that the $\perp$ rule can be derived from either of the two other alternatives. Thus we assume that there is a proof of the form

$$\begin{array}{c} \neg F \\ \vdots \\ \perp \end{array}$$

and we need to show that we then can also prove $F$.

- $$\dfrac{\dfrac{}{F \vee \neg F}\,\text{(law of excluded middle)} \qquad [F]^3 \qquad \dfrac{\begin{array}{c}[\neg F]^3 \\ \vdots \\ \perp\end{array}}{F}\,\perp E}{F}\,\vee E\ (3)$$

- $$\dfrac{\dfrac{\begin{array}{c}[\neg F]^3 \\ \vdots \\ \perp\end{array}}{\neg\neg F}\,\neg I\ (3)}{F}\,\text{(double negation elimination)}$$

**Homework 11.3. [Classical Reasoning (2)]** (5 points)
Assume that the calculus of natural deduction is *augmented* with the two rules from the last exercise. Show:

- $\vdash_N (\neg G \to \neg F) \to (F \to G)$
- $\vdash_N (\neg F \to G) \to (F \vee G)$

*Hint*: You can also use the law of excluded middle in the following form:

$$\frac{\begin{matrix}[F] & [\neg F] \\ \vdots & \vdots \\ G & G\end{matrix}}{G}$$

**Solution:**

- $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(\neg G \to \neg F)]^3 \quad [\neg G]^2}{\neg F} \to E \qquad [F]^1}{\bot} \neg E}{G} \bot\ (2)}{(F \to G)} \to I\ (1)}{(\neg G \to \neg F) \to (F \to G)} \to I\ (3)$$

- $$\dfrac{\dfrac{\dfrac{\dfrac{[\neg F]^1 \quad [\neg F \to G]^3}{G} \neg E}{F \vee G} \vee I_2 \qquad \dfrac{[F]^1}{F \vee G} \vee I_1}{F \vee G} \text{law of excluded middle (1)}}{(\neg F \to G) \to (F \vee G)} \to I\ (3)$$

**Homework 11.4. [Left-Sided Sequent Calculus]** (6 points)
We want to study a modified sequent calculus where the right-hand side is always empty, i.e. where sequents are of the form $\Gamma \Rightarrow$. Give a set of rules for this calculus such that your calculus fulfills the following property and sketch a proof:

$$\Gamma \Rightarrow \Delta \text{ iff } \Gamma, \neg\Delta \Rightarrow$$

*Hint*: Use induction over the length of the derivations. You can skip the cases for $\vee$ and $\wedge$ and instead look at $\to$.

**Solution:** To construct a correct set of rules, one should use the inituition that $\Gamma \Rightarrow$ means that we want to show a contradiction from $\Gamma$. This directly gives us the new axioms:

$$\frac{}{F, \neg F, \Gamma \Rightarrow} \qquad\qquad \frac{}{\bot, \Gamma \Rightarrow}$$

The former 'left' rules then still look very much the same:

$$\frac{F, G, \Gamma \Rightarrow}{F \land G, \Gamma \Rightarrow} \land L' \qquad \frac{F, \Gamma \Rightarrow \quad G, \Gamma \Rightarrow}{F \lor G, \Gamma \Rightarrow} \lor L' \qquad \frac{\neg F, \Gamma \Rightarrow \quad G, \Gamma \Rightarrow}{F \to G, \Gamma \Rightarrow} \to L'$$

For 'right', one way is to look at negated versions of the corresponding operators:

$$\frac{\neg F, \Gamma \Rightarrow \quad \neg G, \Gamma \Rightarrow}{\neg(F \land G), \Gamma \Rightarrow} \land R' \qquad \frac{\neg F, \neg G, \Gamma \Rightarrow}{\neg(F \lor G), \Gamma \Rightarrow} \lor R' \qquad \frac{F, \neg G, \Gamma \Rightarrow}{\neg(F \to G), \Gamma \Rightarrow} \to R'$$

Considering that we have defined negation as a derived operator for sequent calculus in the lecture, the set of 'right' rules is somewhat peculiar. An easy way to avoid trouble is to add a rule to eliminate double negations:

$$\frac{\neg\neg F, \Gamma}{F, \Gamma}$$

Note that the "reversed" rule can already be derived from $\to R'$ (if we allow $\top$ to be dropped anywhere).

We first show that if we have a derivation of length $n$ for $\Gamma \Rightarrow \Delta$ for all $\Gamma$, $\Delta$, then we also have a derivation for $\Gamma, \neg\Delta \Rightarrow$.

Base case $n = 0$: If the $\bot$ axiom was used, we can directly replay the proof. Otherwise, the derivation looks like (with $\Gamma = \{F\} \cup \Gamma'$, $\Delta = \{F\} \cup \Delta'$):

$$\frac{}{F, \Gamma' \Rightarrow F, \Delta'}$$

Again, we directly get the following by the new axiom:

$$\frac{}{F, \Gamma', \neg F, \neg\Delta' \Rightarrow}$$

And thus:

$$\frac{}{\Gamma, \Delta \Rightarrow}$$

Induction step: The induction hypothesis is that for all $\Gamma$ and $\Delta$ if we have a proof of length $n$ for $\Gamma \Rightarrow \Delta$, then we have a proof for $\Gamma, \neg\Delta \Rightarrow$. We are further given a proof for $\Gamma \Rightarrow \Delta$ of length $n + 1$. We consider the last rule used and only look at the two rules for implication:

(Case $\to L$) The proof looks like:

$$\frac{\Gamma' \Rightarrow F, \Delta \quad \overset{\vdots\ \text{length } n}{G, \Gamma' \Rightarrow \Delta}}{F \to G, \Gamma' \Rightarrow \Delta} \to L$$

Using the induction hypothesis two times, we get:

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma', \neg F, \neg\Delta \Rightarrow \quad G, \Gamma', \neg\Delta \Rightarrow\end{array}}{F \to G, \Gamma', \neg\Delta \Rightarrow} \to L'$$

(Case $\to R$) The proof looks like:

$$\dfrac{\begin{array}{c}\vdots \text{ length } n \\ \Gamma, F \Rightarrow G, \Delta'\end{array}}{\Gamma \Rightarrow F \to G, \Delta'} \to R$$

Using the induction hypothesis once, we get:

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma, F, \neg G, \neg\Delta' \Rightarrow\end{array}}{\Gamma, \neg(F \to G, ), \neg\Delta' \Rightarrow} \to R'$$

The other direction: Base case $n = 0$: If the $\perp$ axiom was used, we can directly replay the proof. Otherwise $\{F, \neg F\} \subseteq \Gamma \cup \neg\Delta$ for some $F$. This gives us four cases to consider. The case where $F \in \Gamma$ and $Fin\Delta$ where $\neg F \in \Gamma$ and $\neg Fin\Delta$ folow directly. For the two other cases, we can get a proof for $\Gamma \Rightarrow \Delta$ by first using the axiom for formulas and then one of the rules for negations.

Induction step: The induction hypothesis is that for all $\Gamma$ and $\Delta$ if we have a proof of length $n$ for $\Gamma, \neg\Delta \Rightarrow$, then we have a proof for $\Gamma \Rightarrow \Delta$. We are further given a proof for $\Gamma, \neg\Delta \Rightarrow$ of length $n+1$. We consider the last rule used and only look at the two rules for implication:

(Case $\to L'$) The proof looks like:

$$\dfrac{\begin{array}{c}\vdots \text{ length } n \\ \Gamma', \neg F, \neg\Delta \Rightarrow \quad G, \Gamma', \neg\Delta \Rightarrow\end{array}}{F \to G, \Gamma', \neg\Delta \Rightarrow} \to L'$$

Using the induction hypothesis two times, we get:

$$\dfrac{\begin{array}{c}\vdots \\ \Gamma' \Rightarrow F, \Delta \quad G, \Gamma' \Rightarrow \Delta\end{array}}{F \to G, \Gamma' \Rightarrow \Delta} \to L$$

(Case $\to R'$) We now have to consider two cases because $\neg(F \to G)$ might arise from $\Gamma$ or $\Delta$. The proof can look like:

$$\dfrac{\begin{array}{c}\vdots \text{ length } n \\ \Gamma, F, \neg G, \neg\Delta' \Rightarrow\end{array}}{\Gamma, \neg(F \to G, ), \neg\Delta' \Rightarrow} \to R'$$

Using the induction hypothesis once, we get:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma, F \Rightarrow G, \Delta'\end{array}}{\Gamma \Rightarrow F \to G, \Delta'} \to R$$

And the proof can look like:

$$\dfrac{\begin{array}{c}\vdots \text{ length } n\\ \Gamma', F, \neg G, \neg\Delta \Rightarrow\end{array}}{\Gamma', \neg(F \to G, ), \neg\Delta \Rightarrow} \to R'$$

Using the induction hypothesis and $\neg L$ once, we get:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma', F \Rightarrow G, \Delta\end{array}}{\Gamma' \Rightarrow F \to G, \Delta} \to R}{\Gamma', \neg(F \to G) \Rightarrow \Delta} \neg L$$