

**Exercise 1.1. [Short Questions]**

Let  $M$  be a set of formulas, and let  $F$  and  $G$  be formulas. Which of the following assertions hold?

1. If  $F$  satisfiable then  $M \models F$   $\times$   $F = A_1$   $M = \{A_2\}$
2.  $F$  is valid iff  $\top \models F$   $\checkmark$
3. If  $\models F$  then  $M \models F$   $\checkmark$
4. If  $M \models F$  then  $M \cup \{G\} \models F$   $\checkmark$
5.  $M \models F$  and  $M \models \neg F$  cannot hold simultaneously  $M = \{ \perp \}$
6. If  $M \models G \rightarrow F$  and  $M \models G$  then  $M \models F$   $\checkmark$

$M \models F$   $\iff$  for all  $A$  for all  $M_i \in M, A \models M_i$  implies  $A \models F$   
 $\iff$  for all  $A$  for all  $M_i \in M, A \models M_i$  implies  $A \models (G \rightarrow F) \wedge G$   
 $\iff$   $A \models M_i$  implies  $A \models G \rightarrow F$  and  $A \models G$   
 $\iff$  for all  $A$  for all  $M_i \in M, A \models G \rightarrow F$  and for all  $A$  for all  $M_i \in M, A \models G$   
 $\iff M \models G \rightarrow F$  and  $M \models G$

### Exercise 1.2. [Coincidence Lemma]

Assume that for all atomic formulas  $A_i$  in  $F$ ,  $\mathcal{A}(A_i) = \mathcal{A}'(A_i)$ . Show that

$$\mathcal{A} \models F \text{ iff } \mathcal{A}' \models F$$

Induction over structure  $\models$

- Case  $F = A_i$  for some  $A_i$ :

$$\mathcal{A} \models A_i \quad \text{iff} \quad \mathcal{A}(A_i) = 1 \quad \text{iff} \quad \mathcal{A}'(A_i) = 1 \quad \text{iff} \quad \mathcal{A}' \models A_i$$

- Case  $F = \neg G$  for some  $G$ . Assume  $\mathcal{A} \models G \text{ iff } \mathcal{A}' \models G$ . (IH)

$$\mathcal{A} \models \neg G \quad \text{iff} \quad \mathcal{A} \not\models G \quad \text{iff} \quad \mathcal{A}' \not\models G \quad (\text{by IH})$$

$$\text{iff } \mathcal{A}' \models \neg G$$

- Case  $F = G \vee H$ . Assume  $\mathcal{A} \models G \text{ iff } \mathcal{A}' \models G$

$$\text{and } \mathcal{A} \models H \text{ iff } \mathcal{A}' \models H$$

$$\mathcal{A} \models G \vee H \quad \text{iff} \quad \mathcal{A} \models G \quad \text{or} \quad \mathcal{A} \models H$$

$$\text{iff } \mathcal{A}' \models G \quad \text{or} \quad \mathcal{A}' \models H \quad (\text{by IH})$$

$$\text{iff } \mathcal{A}' \models G \vee H$$

- Case  $F = G \wedge H$  analogous

**Exercise 1.3. [Semantic Proof]**

Let  $\models F \rightarrow G$  where  $F$  and  $G$  do not share any atoms. Show that then either  $F$  is unsatisfiable or  $G$  is a tautology (or both). *Hint: you may want to use the previous result.*

$$\begin{aligned}
 \models F \rightarrow G & \text{ iff } \models \neg F \text{ or } \models G \\
 \parallel & \\
 \models \neg F \text{ or } \models G & \\
 \models F \rightarrow G & \text{ iff } \models \neg F \vee G \quad \left( \begin{array}{l} \text{atom appearing in } F \\ \text{Let } A_F: M_F \rightarrow \{0,1\} \text{ and } A_G: M_G \rightarrow \{0,1\} \\ \text{such that } A_F \cup A_G = A, \text{ for some } A \end{array} \right) \\
 & \text{ iff for all } A, A \models \neg F \vee G \\
 & \text{ iff for all } A, A \models \neg F \text{ or } A \models G \\
 & \text{ iff for all } A, A \models \neg F \text{ or } A \models G \\
 & \text{ iff for all } A, A_F \models \neg F \text{ or } A_G \models G \\
 & \text{ iff for all } A \Rightarrow A'_F \models \neg F \text{ or for all } A'_G, A'_G \models G \\
 & \text{ iff for all } A, A_F \cup A_G \models \neg F \text{ or for all } A, A_F \cup A_G \models G \\
 & \text{ iff } \models \neg F \text{ or } \models G
 \end{aligned}$$

### Exercise 1.4. [Satisfiability Algorithms]

Check the following formulas for satisfiability using one of the algorithms seen in the lecture:

$$\neg(A \wedge D)$$

- $F_1 = (\neg A \vee \neg D \vee B) \wedge D \wedge \neg B \wedge E \wedge (\neg D \vee \neg E \vee C)$

Here Formula 1 CNF

No clause (i.e.  $A \vee B$ )

↳ At most one positive literal in each clause

Rewrite:

$$F_1 \equiv (A \wedge D \rightarrow B) \wedge D \wedge \neg B \wedge E \wedge (D \wedge E \rightarrow C) \\ \wedge (T \rightarrow D) \wedge (B \rightarrow \perp) \wedge (T \rightarrow E)$$

Unit Prop

Mark D, E

Mark C.

RETURN SAT

$$A(B) = A(A) = 0$$

$$A(C) = A(D) = A(E) = 1$$

Satisfying Assignment

$$F_2 = (A \rightarrow C) \wedge (C \rightarrow E) \wedge (\top \rightarrow B) \wedge (C \wedge A \rightarrow D) \wedge (B \rightarrow A) \wedge \\ (B \rightarrow E) \wedge (D \wedge E \rightarrow \perp)$$

Mark  $\beta$  <sup>Unit</sup>

Mark  $A, E$

Mark  $C$

Mark  $D$

Return UNSAT

$$F_3 = (A \rightarrow E) \wedge (B \rightarrow \perp) \wedge (C \rightarrow B) \wedge (T \rightarrow A) \wedge (E \wedge B \rightarrow C) \wedge (C \rightarrow D)$$

Mark  $A$

$$A(A) = A(E) = 1$$

Mark  $E$

$$A(B) = A(C) = A(D) = 0$$

Return SAT

← Any model has to contain this

↑  
For the unmarked variables there may be variability,  
but the algorithm errors that this assignment is satisfying  
if it returns "SAT".