

# Propositional Logic

## Compactness

# Compactness Theorem

Theorem

*A set  $\mathbf{S}$  of formulas is satisfiable  
iff every finite subset of  $\mathbf{S}$  is satisfiable.*

Equivalent formulation:

*A set  $\mathbf{S}$  of formulas is unsatisfiable  
iff some finite subset of  $\mathbf{S}$  is unsatisfiable.*

# Proof

$\Rightarrow$ : If  $\mathbf{S}$  is satisfiable then every finite subset of  $\mathbf{S}$  is satisfiable.

Trivial.

$\Leftarrow$  : If every finite subset of  $\mathbf{S}$  is satisfiable then  $\mathbf{S}$  is satisfiable.

We prove that  $\mathbf{S}$  has a model.

For every  $n \geq 1$  let  $\mathbf{S}_n$  be the subset of formulas of  $\mathbf{S}$  containing only the atoms  $A_1, \dots, A_n$ .

Observe: We have  $\mathbf{S}_1 \subseteq \mathbf{S}_2 \subseteq \mathbf{S}_3 \dots$

# Proof

**Claim 1** Each of the sets  $\mathbf{S}_n$  has a model  $\mathcal{A}_n$ .

**Proof** Partition  $\mathbf{S}_n$  into equivalence classes containing equivalent formulas. There are at most  $2^{2^n}$  classes (**why?**). Pick a representative from each class. The set of all representatives is finite, and so by hypothesis it has a model  $\mathcal{A}_n$ , which is also a model of  $\mathbf{S}_n$ .

**Claim 2**  $\mathcal{A}_n$  is model not only of  $\mathbf{S}_n$ , but also of  $\mathbf{S}_1, \dots, \mathbf{S}_{n-1}$ .

**Proof** follows immediately from  $\mathbf{S}_1 \subseteq \mathbf{S}_2 \subseteq \mathbf{S}_3 \dots$

# Proof

We define a model  $\mathcal{A}$  of  $\mathbf{S}$  by an iterative procedure whose  $n$ -th iteration fixes  $\mathcal{A}(A_n)$ .

$I$  is a set of indices. Step 0:  $I := \mathbb{N}$ .

Step  $n \geq 1$ :

**if** there are infinitely many  $i \in I$  such that  $\mathcal{A}_i(A_n) = 1$

**then**  $\mathcal{A}(A_n) := 1$ ;

$I := I - \{i \mid \mathcal{A}_i(A_n) \neq 1\}$

**else**  $\mathcal{A}(A_n) := 0$ ;

$I := I - \{i \mid \mathcal{A}_i(A_n) \neq 0\}$

**Claim 3** After  $n$  steps,  $n \geq 0$ :

▶  $I$  is infinite

▶  $\mathcal{A}(A_1) = \mathcal{A}_i(A_1), \dots, \mathcal{A}(A_n) = \mathcal{A}_i(A_n)$  for all  $i \in I$

**Proof** by induction on  $n$ .

# Proof

**Claim 4**  $\mathcal{A}(F) = 1$  for every  $F \in \mathbf{S}$ .

**Proof** Let  $F$  contain only  $A_1, \dots, A_j$ .

Therefore  $F \in \mathbf{S}_j \subseteq \mathbf{S}_{j+1} \subseteq \dots$  and  $\mathcal{A}_j(F) = \mathcal{A}_{j+1}(F) = \dots = 1$ .

After step  $j$ , there is some  $i \in I$  s.t.  $i > j$  because  $I$  is infinite.

Because  $\mathcal{A}_i(F) = 1$  and because  $\mathcal{A}$  and  $\mathcal{A}_i$  agree on  $A_1, \dots, A_j$ ,  
 $\mathcal{A}(F) = 1$ .

## Warning

The proof of the compactness lemma is not constructive!

The model  $\mathcal{A}$  can (in general) not be constructed effectively.

The proof only tells us that it exists.

## Corollary

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*If  $\mathbf{S} \models F$  then there is a finite subset  $M \subseteq \mathbf{S}$  such that  $M \models F$ .*