

First-order Predicate Logic

Herbrand Theory

Herbrand universe

The **Herbrand universe** $T(F)$ of a closed formula F in Skolem form is the set of all terms that can be constructed using the function symbols in F .

In the special case that F contains no constants, we first pick an arbitrary constant, say a , and then construct the terms.

Formally, $T(F)$ is inductively defined as follows:

- ▶ All constants occurring in F belong to $T(F)$;
if no constant occurs in F , then $a \in T(F)$
where a is some arbitrary constant.
- ▶ For every n -ary function symbol f occurring in F ,
if $t_1, t_2, \dots, t_n \in T(F)$ then $f(t_1, t_2, \dots, t_n) \in T(F)$.

Note: All terms in $T(F)$ are variable-free by construction!

Example

$$F = \forall x \forall y P(f(x), g(c, y))$$

Herbrand structure

Let F be a closed formula in Skolem form.

A structure \mathcal{A} suitable for f is a **Herbrand structure** for F if it satisfies the following conditions:

- ▶ $U_{\mathcal{A}} = T(F)$, and
- ▶ for every n -ary function symbol f occurring in F and every $t_1, \dots, t_n \in T(F)$: $f^{\mathcal{A}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$.

Fact

If \mathcal{A} is a Herbrand structure, then $\mathcal{A}(t) = t$ for all $t \in U_{\mathcal{A}}$.

Matrix of a formula

Definition

The **matrix** of a formula F is the result of removing all quantifiers (all $\forall x$ and $\exists x$) from F . The matrix is denoted by F^* .

Fundamental theorem of predicate logic

Theorem

Let F be a closed formula in Skolem form.

Then F is satisfiable iff it has a Herbrand model.

Proof If F has a Herbrand model then it is satisfiable.

For the other direction let \mathcal{A} be an arbitrary model of F .

We define a Herbrand structure \mathcal{T} as follows:

Universe $U_{\mathcal{T}} = T(F)$

Function symbols $f^{\mathcal{T}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$

If F contains no constant: $a^{\mathcal{A}} = u$ for some arbitrary $u \in U_{\mathcal{A}}$

Predicate symbols $(t_1, \dots, t_n) \in P^{\mathcal{T}}$ iff $(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n)) \in P^{\mathcal{A}}$

Claim: \mathcal{T} is also a model of F .

Claim: \mathcal{T} is also a model of F .

We prove a stronger assertion:

*For every closed formula G in Skolem form
such that all fun. and pred. symbols in G^* occur in F^* :
if $\mathcal{A} \models G$ then $\mathcal{T} \models G$*

Proof By induction on the number n of universal quantifiers of G .

Basis $n = 0$. Then G has no quantifiers at all.

Therefore $\mathcal{A}(G) = \mathcal{T}(G)$ (why?), and we are done.

Induction step: $G = \forall x H$.

$\mathcal{A} \models G$

\Rightarrow for every $u \in U_{\mathcal{A}}$: $\mathcal{A}[u/x](H) = 1$

\Rightarrow for every $u \in U_{\mathcal{A}}$ of the form $u = \mathcal{A}(t)$
where $t \in T(G)$: $\mathcal{A}[u/x](H) = 1$

\Rightarrow for every $t \in T(G)$: $\mathcal{A}[\mathcal{A}(t)/x](H) = 1$

\Rightarrow for every $t \in T(G)$: $\mathcal{A}(H[t/x]) = 1$ (substitution lemma)

\Rightarrow for every $t \in T(G)$: $\mathcal{T}(H[t/x]) = 1$ (induction hypothesis)

\Rightarrow for every $t \in T(G)$: $\mathcal{T}[\mathcal{T}(t)/x](H) = 1$ (substitution lemma)

\Rightarrow for every $t \in T(G)$: $\mathcal{T}[t/x](H) = 1$ (\mathcal{T} is Herbrand structure)

\Rightarrow $\mathcal{T}(\forall x H) = 1$ ($U_{\mathcal{T}} = T(G)$)

\Rightarrow $\mathcal{T} \models G$

Theorem

Let F be a closed formula in Skolem form.

Then F is satisfiable iff it has a Herbrand model.

What goes wrong if F is not closed or not in Skolem form?

Herbrand expansion

Let $F = \forall y_1 \dots \forall y_n F^*$ be a closed formula in Skolem form.
The **Herbrand expansion** of F is the set of atomic formulas

$$E(F) = \{F^*[t_1/y_1] \dots [t_n/y_n] \mid t_1, \dots, t_n \in T(F)\}$$

Informally: the formulas of $E(F)$ are the result of substituting terms from $T(F)$ for the variables of F^* in every possible way.

Example

$$E(\forall x \forall y P(f(x), g(c, y))) =$$

Note The Herbrand expansion can be viewed as a set of propositional formulas.

Gödel-Herbrand-Skolem Theorem

Theorem

Let F be a closed formula in Skolem form.

Then F is satisfiable iff its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof By the fundamental theorem, it suffices to show:
 F has a Herbrand model iff $E(F)$ is satisfiable.

Let $F = \forall y_1 \dots \forall y_n F^*$.

\mathcal{A} is a Herbrand model of F

iff for all $t_1, \dots, t_n \in T(F)$, $\mathcal{A}[t_1/y_1] \dots [t_n/y_n](F^*) = 1$

iff for all $t_1, \dots, t_n \in T(F)$, $\mathcal{A}(F^*[t_1/y_1] \dots [t_n/y_n]) = 1$

iff for all $G \in E(F)$, $\mathcal{A}(G) = 1$

iff \mathcal{A} is a model of $E(F)$

Herbrand's Theorem

Theorem

Let F be a closed formula in Skolem form.

F is unsatisfiable iff some finite subset of $E(F)$ is unsatisfiable.

Proof Follows immediately from the Gödel-Herbrand-Skolem Theorem and the Compactness Theorem.

Gilmore's Algorithm

Let F be a closed formula in Skolem form
and let F_1, F_2, F_3, \dots be an computable enumeration of $E(F)$.

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Input:  $F$   
 $n := 0$ ;  
repeat  $n := n + 1$ ;  
until  $(F_1 \wedge F_2 \wedge \dots \wedge F_n)$  is unsatisfiable;  
return "unsatisfiable"
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The algorithm terminates iff F is unsatisfiable.

Semi-decidability Theorems

Theorem

- (a) *The unsatisfiability problem of predicate logic is (only) semi-decidable.*
- (b) *The validity problem of predicate logic is (only) semi-decidable.*

Proof

(a) The problem is undecidable.

Gilmore's algorithm is a semi-decision procedure.

(b) F valid iff $\neg F$ unsatisfiable.

Löwenheim-Skolem Theorem

Theorem

Every satisfiable formula of first-order predicate logic has a model with a countable universe.

Proof Let F be a formula, and let G be an equisatisfiable formula in Skolem form (as produced by the Normal Form transformations).

Then for every set S :

F has a model with universe S iff G has a model with universe S .

F satisfiable \Rightarrow G satisfiable
 \Rightarrow G has a Herbrand model (T, I_1)
 \Rightarrow F has a model (T, I_2)
 \Rightarrow F has a countable model
(Herbrand universes are countable)

Löwenheim-Skolem Theorem

Formulas of first-order logic cannot enforce uncountable models

Formulas of first-order logic cannot axiomatize the real numbers
because there will always be countable models