

# Quantifier Elimination

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i.e. one can replace a subterm  $F$  of  $H$  by  $G$ .*

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*Then  $T$  is decidable and complete.*

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Method: Eliminate quantifiers bottom-up by  $qe1$ , use  $\forall \equiv \neg \exists \neg$



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The rest is theory specific.



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Models of DLO?

**Theorem**

*All countable DLOs are isomorphic.*

## Quantifier elimination example

### Example

$$DLO \models \exists y (x < y \wedge y < z) \leftrightarrow$$

## Elimination of “ $\neg$ ”

Elimination of negative literals (function *aneg*):

$$DLO \models \neg x = y \leftrightarrow x < y \vee y < x$$

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3. Separate atoms into lower and upper bounds for  $x$  and use

$$DLO \models \exists x \left( \bigwedge_{i=1}^m l_i < x \wedge \bigwedge_{j=1}^n x < u_j \right) \leftrightarrow \bigwedge_{i=1}^m \bigwedge_{j=1}^n l_i < u_j$$

Special case:  $\bigwedge_{k=1}^0 F_k = \top$

### Examples

$$\exists x (x < z \wedge y < x \wedge x < y') \leftrightarrow ?$$

$$\forall x (x < y) \leftrightarrow ?$$

$$\exists x \exists y \exists z (x < y \wedge y < z \wedge z < x) \leftrightarrow ?$$

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you cannot distinguish models of DLO by first-order formulas.

# Quantifier Elimination

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A term in **normal form**:  $c_1 \cdot x_1 + \dots + c_n \cdot x_n + c$

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Function  $sol_x(A)$  solves  $A$  for  $x$ .

## Elimination of “ $\neg$ ”

Elimination of negative literals (function *aneg*):

$$R_+ \models \neg x = y \leftrightarrow x < y \vee y < x$$

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# Fourier-Motzkin Elimination



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Can DNF be avoided?

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Define  $qe1(\exists x F)$ :

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2. Apply Ferrante and Rackoff's theorem.

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2. Apply Ferrante and Rackoff's theorem.

## Theorem

*Eliminating all quantifiers with Ferrante and Rackoff's procedure from a formula of size  $n$  takes space  $O(2^{cn})$  and time  $O(2^{2^{dn}})$ .*

# Quantifier Elimination

# Linear Integer Arithmetic

See [Harrison] or [Enderton] under “Presburger”