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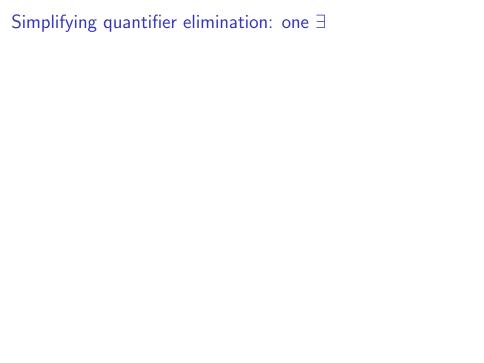
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Quantifier Elimination

Dense Linear Orders

Without Endpoints

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Let DLO stand for "dense linear order without endpoints" and for the following set of axioms:

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Models of DLO?

Theorem

All countable DLOs are isomorphic.

Quantifier elimination example

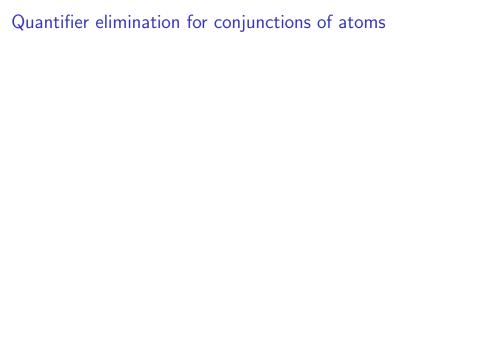
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- ► Th(DLO) has quantifier elimination
- ► Th(DLO) is decidable and complete
- ► All models of DLO (for example (Q, <) and (R, <)) are elementarily equivalent: you cannot distinguish models of DLO by first-order formulas.

Quantifier Elimination

Linear real arithmetic

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Elimination of negative literals (function aneg):

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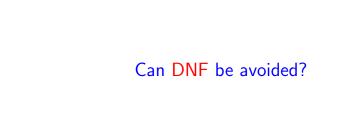
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$$\exists x \forall y \ (3y < x \lor x < 2y) \ \leftrightarrow ?$$





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Eliminating all quantifiers with Ferrante and Rackoff's procedure from a formula of size n takes space $O(2^{cn})$ and time $O(2^{2^{dn}})$.

Quantifier Elimination Linear Integer Arithmetic

See [Harrison] or [Enderton] under "Presburger"