

# First-order Predicate Logic Theories

# Definitions

## Definition

A **signature**  $\Sigma$  is a set of predicate and function symbols.

A  **$\Sigma$ -formula** is a formula that contains only predicate and function symbols from  $\Sigma$ .

A  **$\Sigma$ -structure** is a structure that interprets all predicate and function symbols from  $\Sigma$ .

## Definition

A **sentence** is a closed formula.

In the sequel,  $S$  is a set of sentences.

# Theories

## Definition

A **theory** is a set of sentences  $S$  such that  $S$  is closed under consequence: If  $S \models F$  and  $F$  is closed, then  $F \in S$ .

Let  $S$  be a set of  $\Sigma$ -sentences.

$Mod(S)$  is the class of all models of  $S$ :

$$Mod(S) = \{ \mathcal{A} \mid \mathcal{A} \text{ } \Sigma\text{-structure and for all } F \in S, \mathcal{A} \models F \}$$

Let  $\mathcal{M}$  be a class of  $\Sigma$ -structures:

$Th(\mathcal{M})$  is the set of all sentences true in all structures in  $\mathcal{M}$ :

$$Th(\mathcal{M}) = \{ F \mid F \text{ } \Sigma\text{-sentence and for all } \mathcal{A} \in \mathcal{M}, \mathcal{A} \models F \}$$

# Theories

## Fact

- ▶  $S \subseteq Th(Mod(S))$
- ▶  $\mathcal{M} \subseteq Mod(Th(\mathcal{M}))$
- ▶  $Th(\mathcal{M})$  is a theory
- ▶  $Th(Mod(S)) = \{F \mid F \text{ } \Sigma\text{-sentence and } S \models F\}$

# Examples

## Example (Groups)

$\Sigma = \{e, i, *, =\}$  (where  $e$  is a constant and  $i$  is unary)

$G = \{\forall x \forall y \forall z (x * y) * z = x * (y * z), \forall x i(x) * x = e, \forall x e * x = x\}$

- ▶ Every group is a model of  $G$
- ▶  $Mod(G)$  is the class of all groups
- ▶  $G \subset Th(Mod(G))$

## Examples

**Notation:**  $(\mathbb{Z}, +, \leq)$  denotes the structure with universe  $\mathbb{Z}$  and the standard interpretations for the symbols  $+$  and  $\leq$ .

The same notation is used for other standard structures where the interpretation of a symbol is clear from the symbol.

### Example (Linear integer arithmetic)

$Th(\mathbb{Z}, +, \leq)$  is the set of all sentences over the signature  $\{+, \leq\}$  that are true in the structure  $(\mathbb{Z}, +, \leq)$ .

# Theory of a structure

In general:  $Th(\mathcal{A})$  is short for  $Th(\{\mathcal{A}\})$ .

## Fact

*Let  $\mathcal{A}$  be a  $\Sigma$ -structure and  $F$  a  $\Sigma$ -sentence.*

*Then  $\mathcal{A} \models F$  iff  $Th(\mathcal{A}) \models F$ .*

# Axioms and consequences

## Definition

Let  $S$  be a set of  $\Sigma$ -sentences.

$Cn(S)$  is the set of **consequences** of  $S$ :

$$Cn(S) = \{F \mid F \text{ } \Sigma\text{-sentence and } S \models F\}$$

A theory  $T$  is **axiomatized** by  $S$  if  $T = Cn(S)$

A theory  $T$  is **axiomatizable** if there is some decidable or recursively enumerable  $S$  that axiomatizes  $T$ .

A theory  $T$  is **finitely axiomatizable**

if there is some finite  $S$  that axiomatizes  $T$ .

## Example

$Cn(\emptyset)$  is the set of valid sentences.

$Cn(G)$  is the set of sentences that are true in all groups.



## Famous numerical theories

$Th(\mathbb{R}, +, \leq)$  is called **linear real arithmetic**.

It is decidable.

$Th(\mathbb{R}, +, *, \leq)$  is called **real arithmetic**.

It is decidable.

$Th(\mathbb{Z}, +, \leq)$  is called **linear integer arithmetic** or **Presburger arithmetic**.

It is decidable.

$Th(\mathbb{Z}, +, *, \leq)$  is called **integer arithmetic**.

It is not even **semidecidable** (= r.e.).

Decidability via special algorithms.

# Completeness and elementary equivalence

## Definition

A theory  $T$  is **complete** if for every sentence  $F$ ,  $T \models F$  or  $T \models \neg F$ .

## Definition

Two structures  $\mathcal{A}$  and  $\mathcal{B}$  are **elementarily equivalent** if  $Th(\mathcal{A}) = Th(\mathcal{B})$ .

## Theorem

*A theory  $T$  is complete iff all its models are elementarily equivalent.*