

Propositional Logic

Compactness

Compactness Theorem

Theorem

*A set S of formulas is satisfiable
iff every finite subset of S is satisfiable.*

Equivalent formulation:

*A set S of formulas is unsatisfiable
iff some finite subset of S is unsatisfiable.*

Proof

\Rightarrow : If S is satisfiable then every finite subset of S is satisfiable.

Trivial.

\Leftarrow : If every finite subset of S is satisfiable then S is satisfiable.

We prove that S has a model.

Proof

Terminology: \mathcal{A} is a b_1, \dots, b_n model of T
(where $b_1, \dots, b_n \in \{0, 1\}^*$ and T is a set of formulas)
if $\mathcal{A}(A_i) = b_i$ (for $i = 1, \dots, n$) and $\mathcal{A} \models T$.

Define an infinite sequence b_1, b_2, \dots recursively as follows:

$b_{n+1} =$ some $b \in \{0, 1\}$ s.t.
all finite $T \subseteq S$ have a b_1, \dots, b_n, b model.

Claim 1: For all n , all finite $T \subseteq S$ have a b_1, \dots, b_n model.

Proof by induction on n .

Case $n = 0$: because all finite $T \subseteq S$ are satisfiable.

Step: Need to show that a suitable b exists.

Proof by contradiction. Assume there is no suitable b .

Then there is a finite $T_0 \subseteq S$ that has no $b_1, \dots, b_n, 0$ model (0)

and there is a finite $T_1 \subseteq S$ that has no $b_1, \dots, b_n, 1$ model (1).

Therefore $T_0 \cup T_1$ has no b_1, \dots, b_n model \mathcal{A} :

$\mathcal{A}(A_{n+1}) = 0$ contradicts (0), $\mathcal{A}(A_{n+1}) = 1$ contradicts (1).

But by IH: $T_0 \cup T_1$ has a b_1, \dots, b_n model — Contradiction!

Proof

Define $\mathcal{B}(A_i) = b_i$ for all i .

Claim 2: $\mathcal{B} \models S$

We show $\mathcal{B} \models F$ for all $F \in S$.

Let m be the maximal index of all atoms in F .

By Claim 1, $\{F\}$ has a b_1, \dots, b_m model \mathcal{A} .

Hence $\mathcal{B} \models F$ because \mathcal{A} and \mathcal{B} agree on all atoms in F .

Corollary

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If $S \models F$ then there is a finite subset $M \subseteq S$ such that $M \models F$.